THE COLLEGES OF OXFORD UNIVERSITY

PHYSICS

Wednesday 7 November 2012

Time allowed: 2 hours

For candidates applying for Physics, Physics and Philosophy, Engineering or Materials

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There are two parts (A and B) to this test, carrying equal weight.

Answers should be written on the question sheet in the spaces provided and you should attempt as many questions as you can from each part.

Marks for each question are indicated in the right hand margin. There are a total of 100 marks available and total marks for each section are indicated at the start of a section. You are advised to divide your time according to the marks available, and to spend equal effort on parts A and B.

No calculators, tables or formula sheets may be used.

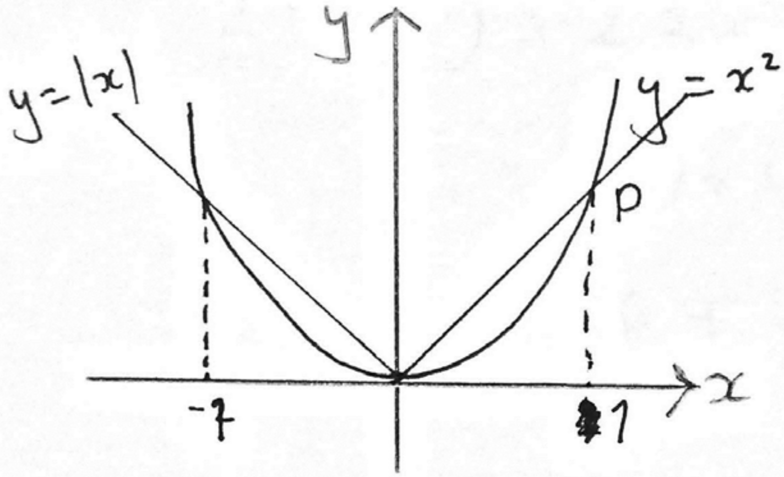
Answers in Part A should be given exactly unless indicated otherwise. Numeric answers in Part B should be calculated to 2 significant figures.

Use $g = 10\,\mathrm{m\,s^{-2}}$.

Do NOT turn over until told that you may do so.

Part A: Mathematics for Physics [50 Marks]

1. Find the area between $y = x^2$ and $y = | x |$. [4]



At P, $x = x^2$

$0 = x^2 - x$

$= x(x-1)$

$x = 0$ or 1

origin P

$$\Rightarrow A = 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= \frac{1}{3} \text{ unit}^2$$

2. (i) Write down the binomial expansion of $(4+x)^4$.
(ii) Hence or otherwise evaluate $(4.2)^4$ to 2 d.p (decimal places). [4]

i) $(4+x)^4 = 4^4 + 4(4)^3 x + 6(4)^2 x^2$
$+ 4(4)x^3 + x^4$
$= \underline{256 + 256x + 96x^2 + 16x^3 + x^4}$

ii) $(4.2)^4 = (4+0.2)^4$
$= 256 + 256(0.2) + 96(0.04)$
$+ 16(0.008) + 0.0016$
$= 256 + 51.2 + 3.84 + 0.128$
$+ 0.0016$
$= \underline{311.17}$ (2dp)

3. Evaluate $\sum_{r=1}^{8}(2+4^r)$. [3]

$$\sum_{r=1}^{8}(2+4^r) = \sum_{r=1}^{8} 2 + \sum_{r=1}^{8} 4^r$$

$$= (8\times 2) + 4 + 4^2 + 4^3 + \dots 4^8$$

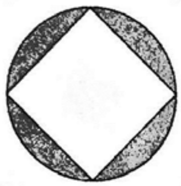
$$= 16 + \frac{4(4^8-1)}{4-1}$$

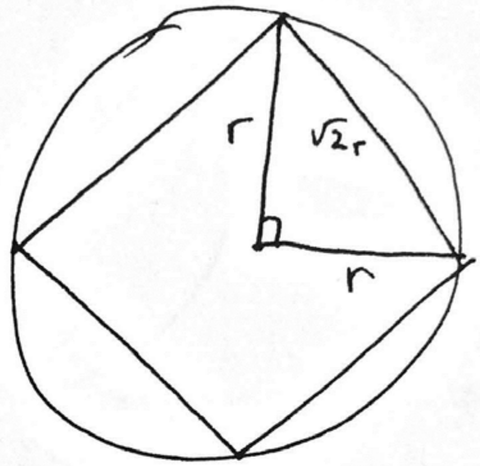
$$= 16 + \frac{4(65535)}{3}$$

$$= 16 + (4\times 21845)$$

$$= \underline{87396}$$

4. Consider a square inside a circle of radius r as shown. What is the shaded area in terms of r? [3]





Area of circle $= \pi r^2$

Area of square $= (\sqrt{2}r)^2 = 2r^2$

Shaded area $= \pi r^2 - 2r^2$

$= \underline{(\pi - 2)r^2}$

5. Show $x = 1$ is a solution to $x^3 - 6x^2 - 9x + 14 = 0$ and find the other solutions. [4]

Let $f(x) = x^3 - 6x^2 - 9x + 14$

$f(1) = 1^3 - 6(1)^2 - 9(1) + 14$
$= 1 - 6 - 9 + 14$
$= 0$

$\therefore x = 1$ is a root of $f(x) = 0$

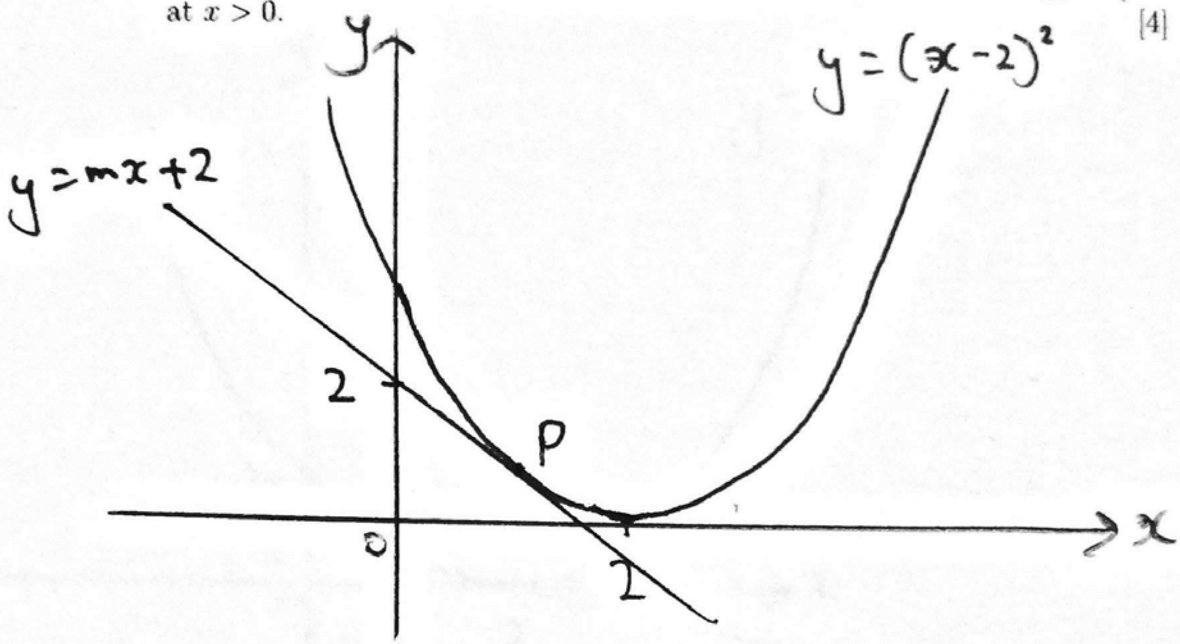
$\Rightarrow (x-1)$ is a factor of $f(x)$

$(x-1)(x^2 - 5x - 14) = x^3 - 6x^2 - 9x + 14 = 0$

$(x-1)(x-7)(x+2) = 0$

$x = 1$ or $\underline{7}$ or $\underline{-2}$

6. Find the equation of the line passing through $(0, 2)$ and just touching $y = (x-2)^2$ at $x > 0$. [4]



At P, $y = mx + 2 = (x-2)^2$

$mx + 2 = x^2 - 4x + 4$

$0 = x^2 - (4+m)x + 2$

Also, gradients are equal at P:

$\frac{dy}{dx} = 2(x-2) = 2x - 4 = m$

$\Rightarrow 0 = x^2 - (4 - 4 + 2x)x + 2$

$= x^2 - 2x^2 + 2$

$x^2 = 2$

$x = \pm\sqrt{2}$

For $x > 0$, $m = 2\sqrt{2} - 4$

$\therefore \underline{y = (2\sqrt{2} - 4)x + 2}$

7. If $5 = \log_2 16 + \log_{10} \sqrt{0.01} + \log_3 x$, determine x. [4]

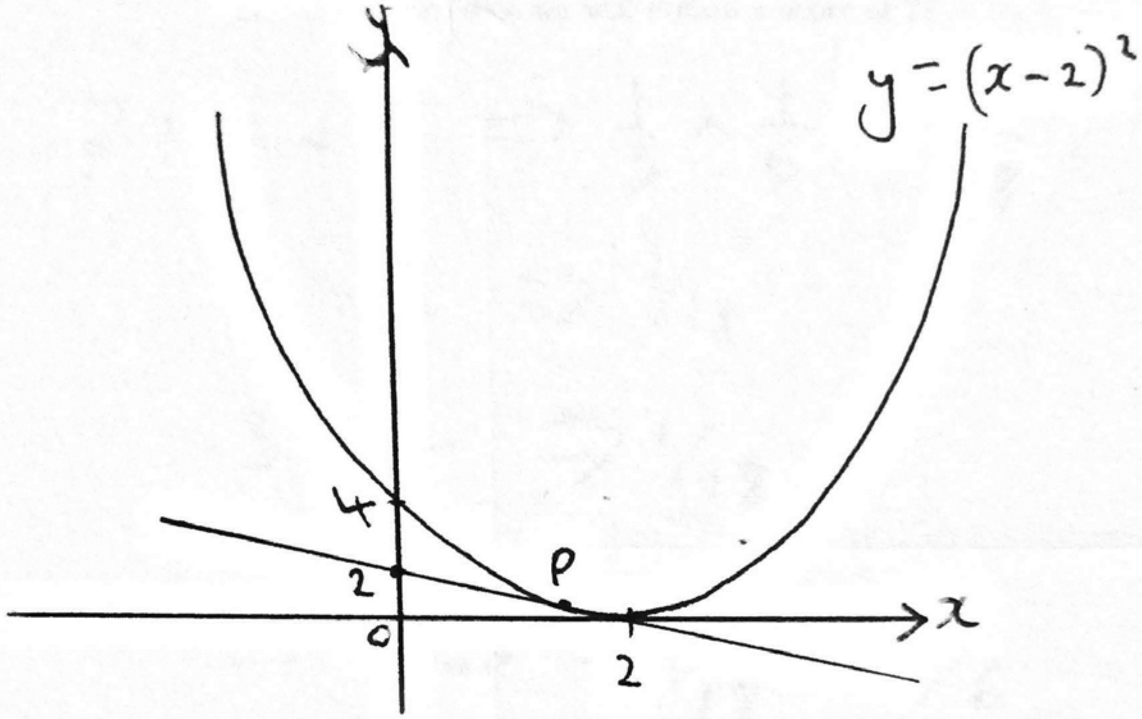
$5 = 4 + \log_{10}(0.1) + \log_3 x$

$1 = -1 + \log_3 x$

$2 = \log_3 x$

$\underline{x = 9}$

6. Find the equation of the line passing through $(0, 2)$ and just touching $y = (x-2)^2$ at $x > 0$. [4]



At P, gradients are same:

$y = (x-2)^2$

$\frac{dy}{dx} = 2(x-2) = 2x-4$

$\Rightarrow$ Eqn. is $y = (2x-4)x + 2$

Also at P, $(x-2)^2 = (2x-4)x + 2$

$x^2 - 4x + 4 = 2x^2 - 4x + 2$

$0 = x^2 - 2$

$x = \pm\sqrt{2}$

But $x > 0$ $\therefore$ $x = \sqrt{2}$

$\Rightarrow$ Eqn. is $y = (2\sqrt{2} - 4)x + 2$

7. If $5 = \log_2 16 + \log_{10} \sqrt{0.01} + \log_3 x$, determine x. [4]

$5 = 4 + \log_{10} 0.1 + \log_3 x$

$5 = 4 - 1 + \log_3 x$

$=$

$2 = \log_3 x$

$\underline{x = 9}$

8. Consider two dice – one contains the numbers 1-6, the other contains only 1,2,3 each shown twice (i.e. 1,2,3,1,2,3). What is the probability that when we roll the two dice we will obtain a score of 7? [4]

$7: \; 1+6 \Rightarrow \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$

$2+5 \Rightarrow 1/18$

$3+4 \Rightarrow 1/18$

$\therefore \text{Prob} = 3/18 = \underline{1/6}$

9. Solve $\cos^2\theta + \sin\theta = 0$ for θ. Leave your answer in terms of $\sin\theta$. [4]

$$1 - \sin^2\theta + \sin\theta = 0$$

$$\sin^2\theta - \sin\theta - 1 = 0$$

Let $\cancel{y}$ $x = \sin\theta$

$$x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$\sin\theta < 1$

$$\Rightarrow \underline{\sin\theta = \frac{1-\sqrt{5}}{2}}$$

10. Sketch an example of a real function $f(x)$ defined for all real arguments x, which has all of the following properties:

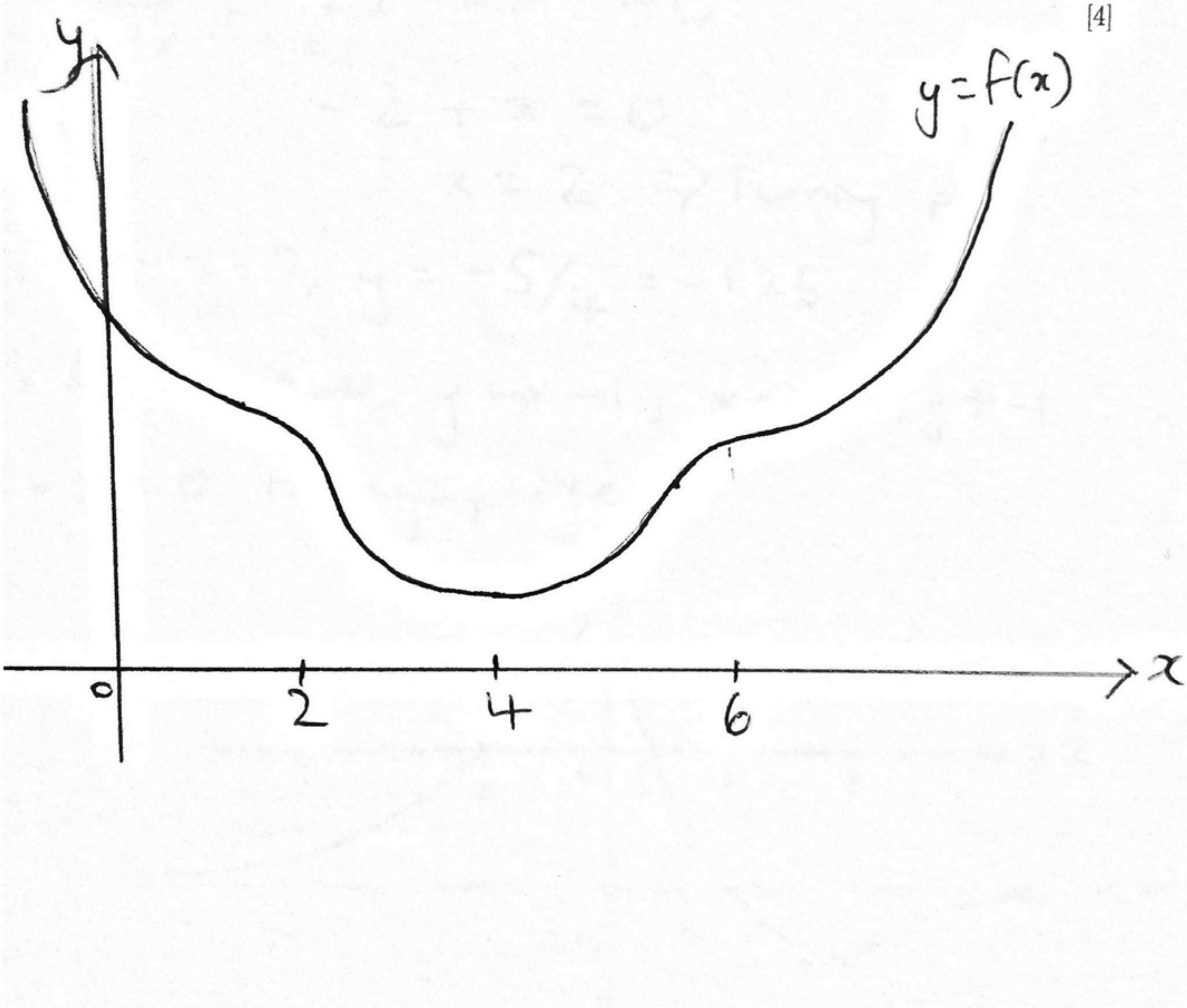
(a) $f(x) > 0$ for all x,

(b) $f(x)$ is a continuous function,

(c) $\frac{df}{dx} = 0$ only for $x = 4$,

(d) $\frac{d^2 f}{dx^2} = 0$ only for $x = 2$ and $x = 6$.

[4]



11. Solve $-1 < -\frac{1}{x} + 2x < 1$. [6]

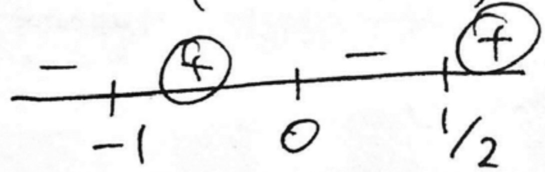
$-1 < -\frac{1}{x} + 2x$ and $-\frac{1}{x} + 2x < 1$

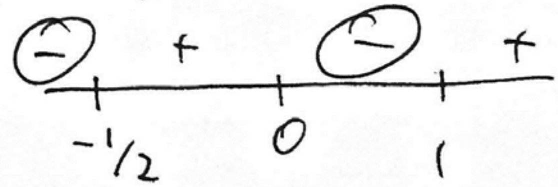
$x^2 < -x + 2x^3$ $\qquad$ $-x + 2x^3 < x^2$

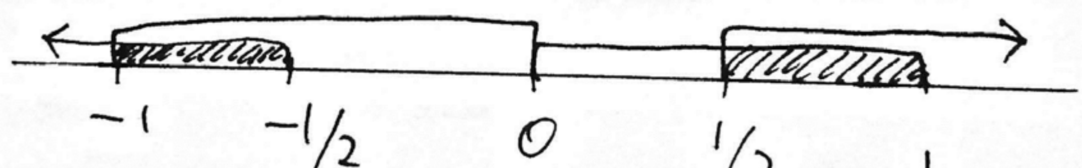
$0 < 2x^3 + x^2 - x$ $\qquad$ $2x^3 - x^2 - x < 0$

$0 < x(2x^2 + x - 1)$ $\qquad$ $x(2x^2 - x - 1) < 0$

$0 < x(2x-1)(x+1)$ $\qquad$ $x(2x+1)(x-1) < 0$







$\underline{\{-1 < x < -1/2\} \cup \{1/2 < x < 1\}}$

12. Sketch $y = \dfrac{1 - x - x^2}{x^2}$. [6]

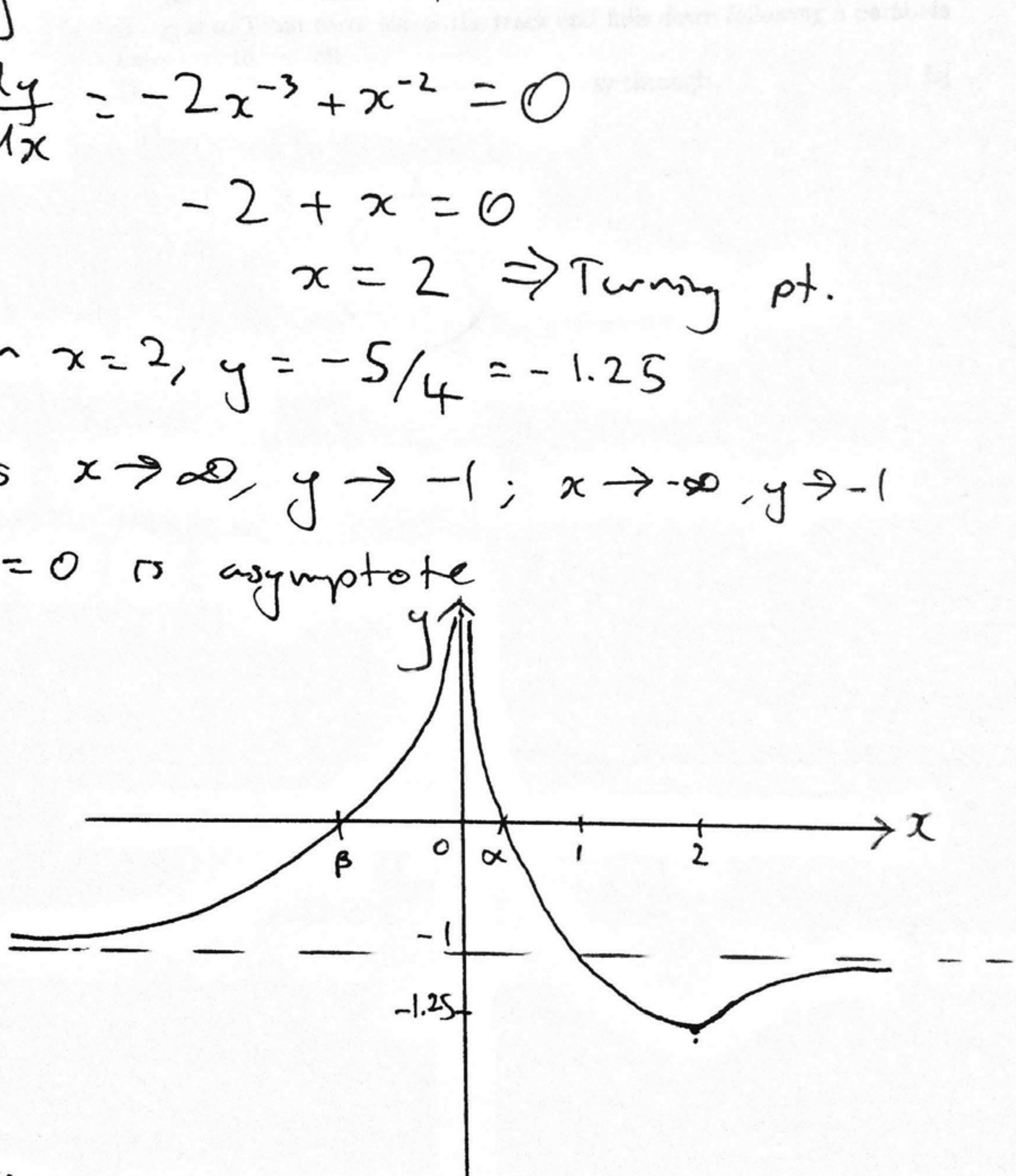
$$y = x^{-2} - x^{-1} - 1$$

$$\frac{dy}{dx} = -2x^{-3} + x^{-2} = 0$$

$$-2 + x = 0$$

$$x = 2 \Rightarrow \text{Turning pt.}$$

- when $x = 2$, $y = -5/4 = -1.25$
- As $x \to \infty$, $y \to -1$; $x \to -\infty$, $y \to -1$
- $x = 0$ is asymptote



- when $y = 0$, $1 - x - x^2 = 0$

$$x = \frac{1 \pm \sqrt{1+4}}{-2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} = \alpha, \beta$$

Part B: Physics [50 Marks]

Multiple choice (10 marks)

Please circle **one** answer to each question only.

13. A vintage steam locomotive made of iron has a mass of 6.5×10^4 kg and is 10 m long. How long is its scale model which is also made out of iron and has a mass of 1 kg?

A $\simeq 4$ cm **B** $\simeq 20$ cm
C $\simeq 25$ cm **D** $\simeq 30$ cm [2]

14. A gas cylinder has a volume of 0.02 m^3 and contains 88 g of carbon dioxide at a temperature of 27° C. The molar gas constant R $\simeq 8.3$ $Jmol^{-1}K^{-1}$. What is the gas pressure?

A $\simeq 101$ kPa **B** $\simeq 149$ kPa
C $\simeq 201$ kPa **D** $\simeq 249$ kPa [2]

15. An electric car has a battery pack delivering 160 V and 100 A of steady current when moving at 36 km/h. What is the air resistance, assuming 100 % efficiency?

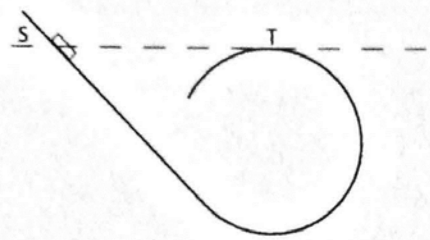
A $\simeq 440$ N **B** $\simeq 1600$ N
C $\simeq 2000$ N **D** $\simeq 3200$ N [2]

16. A cube painted black is cut into 125 identical cubes. How many of them are not painted at all?

A 21 **B** 25
C 27 **D** 30 [2]

17. A massive slider starts from rest from a point S (which is at the same height as a point T at the top of the track) and slides along a frictionless circular track as sketched in figure below. The slider

A does not get to T.
[redacted] gets to T and falls straight down.
C gets to T but then, leaves the track and falls down following a parabola trajectory to the left.
D passes T staying on the track all the way through. [2]



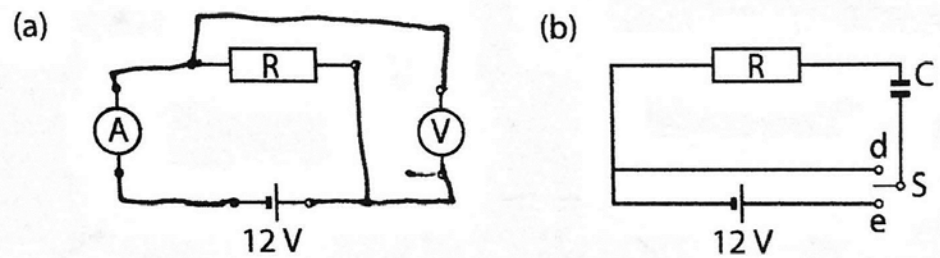
Ans: A

Written answers (20 marks)

18. A 12V battery, a voltmeter, an ammeter and a resistor $R = 2$ kΩ are sketched in figure (a) below.

Sketch connections to create a circuit to measure a potential difference across the resistor and an electric current. How big is the current? [1]

A capacitor $C = 4\ \mu$F and a switch S, as sketched in figure (b), are inserted into the circuit. Sketch how the current depends on time from the moment t_e when the switch is moved to e closing the circuit. Estimate the time T after which the current is not changing significantly. After a time t_d much longer than T, the switch is moved to d. Sketch the current from that moment until the moment when the current is not changing significantly, indicating on your sketch the time interval T. [3]



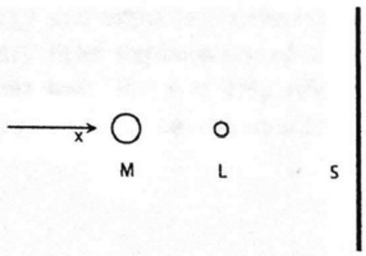
$$I = \frac{V}{R} = \frac{12}{2000} = \underline{6\,mA}$$

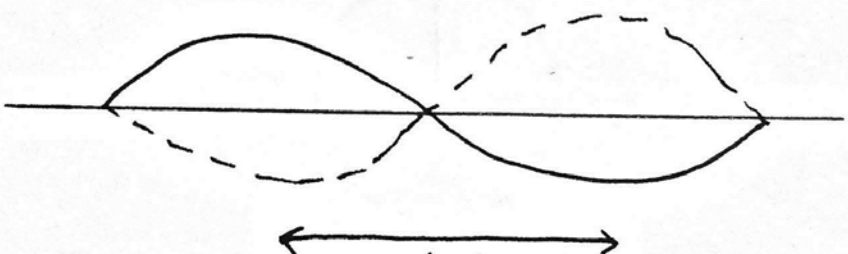
b)

$RC = 2 \times 4 \times 10^{-3} = 8\,ms$

$\Rightarrow \underline{T \approx 40\,ms}$

19. A loudspeaker L is placed between a microphone M and a screen S reflecting sound waves as sketched below. The loudspeaker emits sound of fixed wavelength λ in all directions. When the screen is moving slowly, to the right along the x direction (slowly in comparison with the speed of sound), the microphone records minima and maxima of the sound intensity. What is the distance between two screen positions giving two successive maxima? Would the microphone record minima and maxima if (a) the loudspeaker or (b) the microphone is moving along x direction instead of the screen? [4]





$\lambda/2$

a) Yes b) No

20. The ^{238}U isotope has a half-life $T_{238} = 4.5\text{x}10^9$ years and the ^{235}U has $T_{235} = 7.0\text{x}10^8$ years. $N_{238}(t)$ is the number of ^{238}U nuclei at time t and $N_{235}(t)$ is the corresponding number for ^{235}U. The relative abundance $r(t)$ is defined as $r(t) = \frac{N_{235}(t)}{N_{238}(t)}$. At present, $r = 0.0072$. Estimate the relative abundance of these two isotopes 10^9 years ago. You might use the following approximations: $e^x \simeq 1 + x$ for small x, $e \simeq 2.7$ and $\ln 2 \simeq 0.7$. [4]

$$r(t) = \frac{N_{235}(t)}{N_{238}(t)} = \frac{N^{\circ}_{235}\, e^{-\lambda_{235} t}}{N^{\circ}_{238}\, e^{-\lambda_{238} t}}$$

For $t = -10^9$,

$$0.0072 = \frac{N^{\circ}_{235}\, e^{-\lambda_{235} \times 10^9}}{N^{\circ}_{238}\, e^{-\lambda_{238} \times 10^9}}$$

Using $t_{1/2} = \frac{\ln 2}{\lambda}$ and $\frac{N^{\circ}_{235}}{N^{\circ}_{238}} = r_0$

$$0.0072 = r_0 \frac{e^{-\frac{\ln 2 \times 10^9}{0.7 \times 10^9}}}{e^{-\frac{\ln 2 \times 10^9}{4.5 \times 10^9}}}$$

$$= r_0 \frac{e^{-0.7/0.7}}{e^{-0.7/4.5}}$$

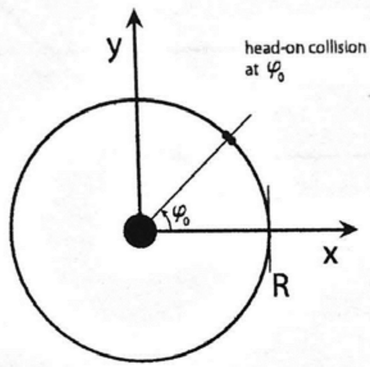
$$= \frac{r_0 (2.7)^{-1}}{1 - \frac{0.7}{4.5}}$$

$$r_0 = 0.0072 \times \frac{38}{45} \times 2.7$$

$$\approx 0.016$$

21. A meteoroid of mass m is on a circular Earth orbit of radius R which is a few (> 2) times larger than the radius of the Earth R_E. Derive an expression for the meteoroid's speed. State the meanings of all symbols used. [2]

Another meteoroid of the same mass is on the same orbit, in the same plane but rotating in the opposite direction. At an azimuthal angle φ_0, see figure below, the two meteoroids collide head-on and coalesce (combine). Sketch the complete trajectory of the newly formed double mass meteoroid showing how the azimuthal angle φ depends on the distance r. Sketch also its kinetic energy and expected meteoroid surface temperature as a function of r. Give a very brief explanation of why you expect the temperature to depend on r that way. For $r < 2R_E$ effects due to the Earth's atmosphere can not be neglected; r is the distance from the Earth's centre. [6]



$$F = ma$$

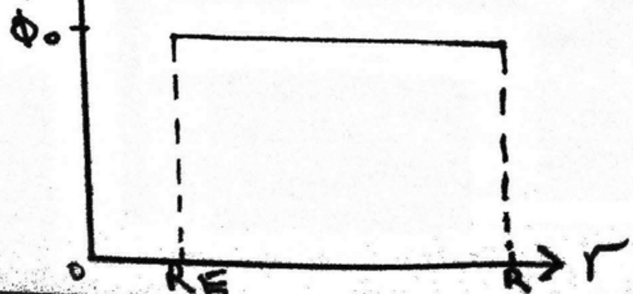
$$\frac{GMm}{R^2} = \frac{mv^2}{R}$$

$$v = \sqrt{\frac{GM}{R}}$$

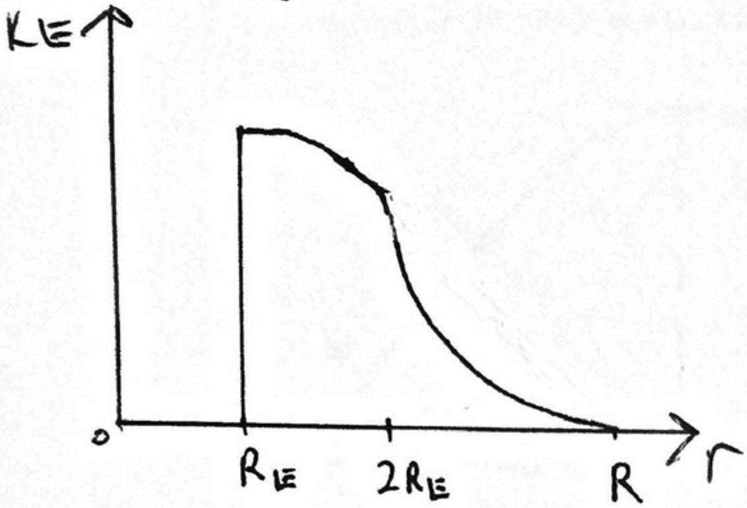
G: Gravitational constant

M: Mass of the Earth

Trajectory, showing Φ against r:



KE against r:

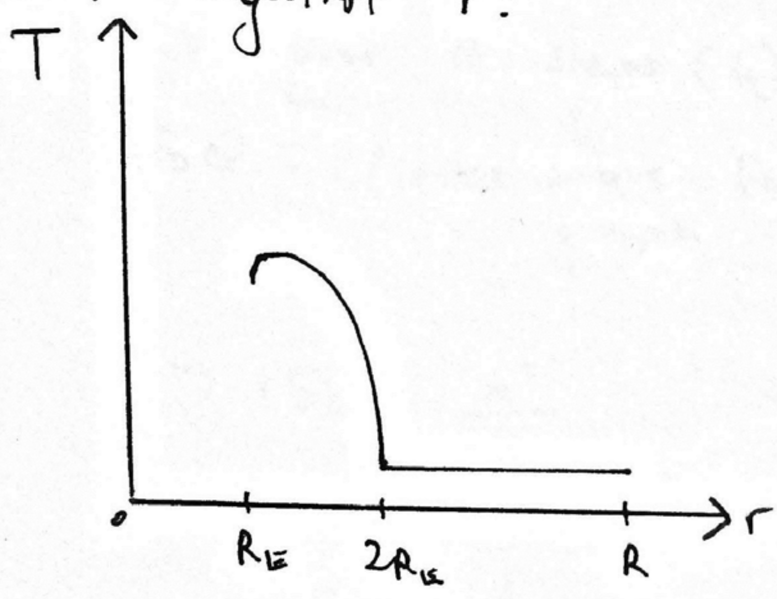


$KE = \frac{1}{2}mv^2$

$GPE = -\frac{GMm}{r}$

$\Rightarrow KE \propto \frac{1}{r}$

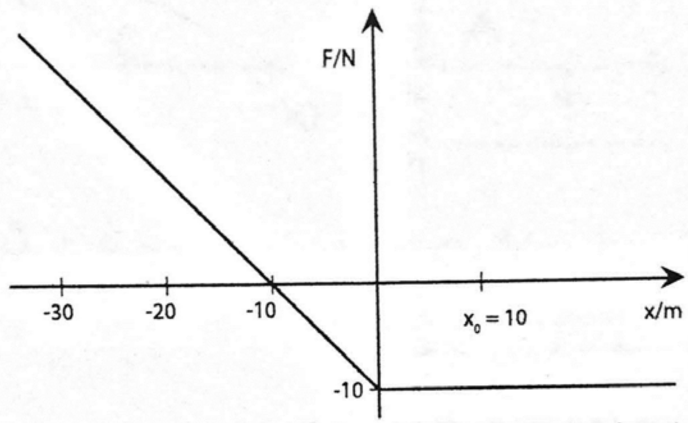
T against r:



There's no heating outside the Earth's atmosphere, causing constant temperature.

Long question (20 marks)

22. A point like object with mass $m = 1$ kg starts from rest at point $x_0 = 10$ m and moves without any friction under a force F which depends on the coordinate x as illustrated in figure below. The motion is confined to one dimension along x.



a1 What is its speed at $x = 0$? [2]

$$\frac{1}{2}mv^2 = Fx$$

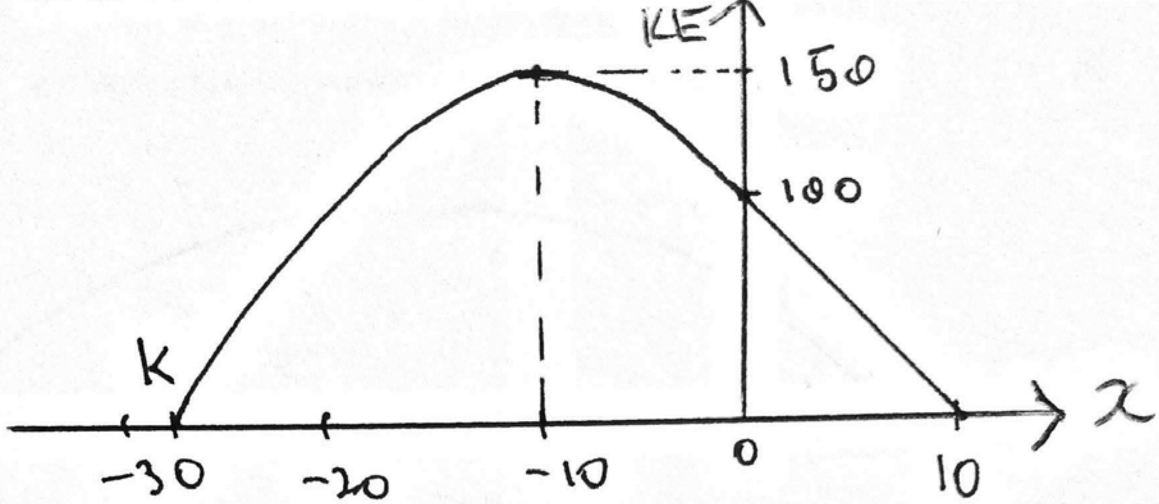
$$v^2 = 2 \times 10 \times 10$$

$$v = \sqrt{200}$$

$$= 10\sqrt{2}$$

$$= \underline{14 ms^{-1}} \ (2sf)$$

a2 Sketch its kinetic energy as a function of x. [4]



Max $KE = (10 \times 10) + (10 \times 10 \times \frac{1}{2}) = 150\,J$

Point where it stops (k):

150 = Area above force-distance graph

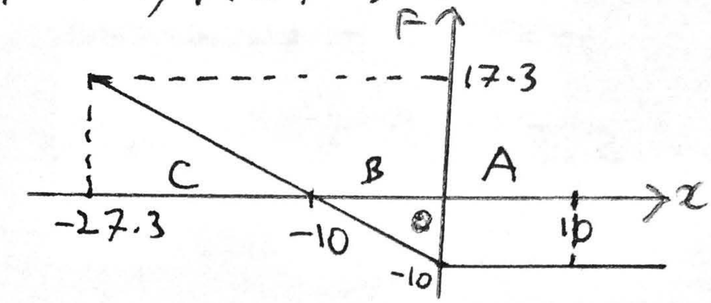
$$\Rightarrow 150 = \frac{a^2}{2}$$

$$a = 17.3\,m$$

$$\therefore k = -27.3\,m$$

a3 Sketch its velocity as well as its acceleration as a function of time t. [6]

$F = ma,\ m = 1 \Rightarrow F = a$



$A: s = -10,\ u = 0,\ v = -14,\ a = -10,\ t = ?$

$v = u + at$

$-14 = -10t$

$t = 1.4s$

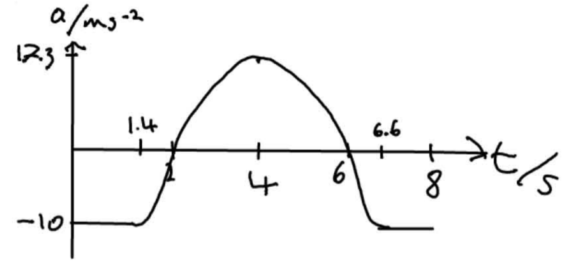
$B: u = 14,\ v = 17.3$

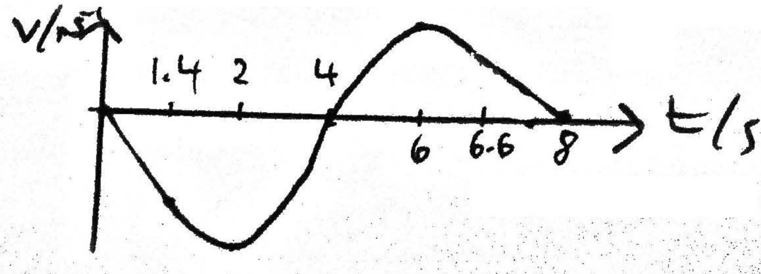
$\frac{1}{2} \times 10 \times t = 17.3 - 14$

$t = 0.64\ s$

$C: \frac{1}{2} \times 17.3 \times t = 17.3 - 0$

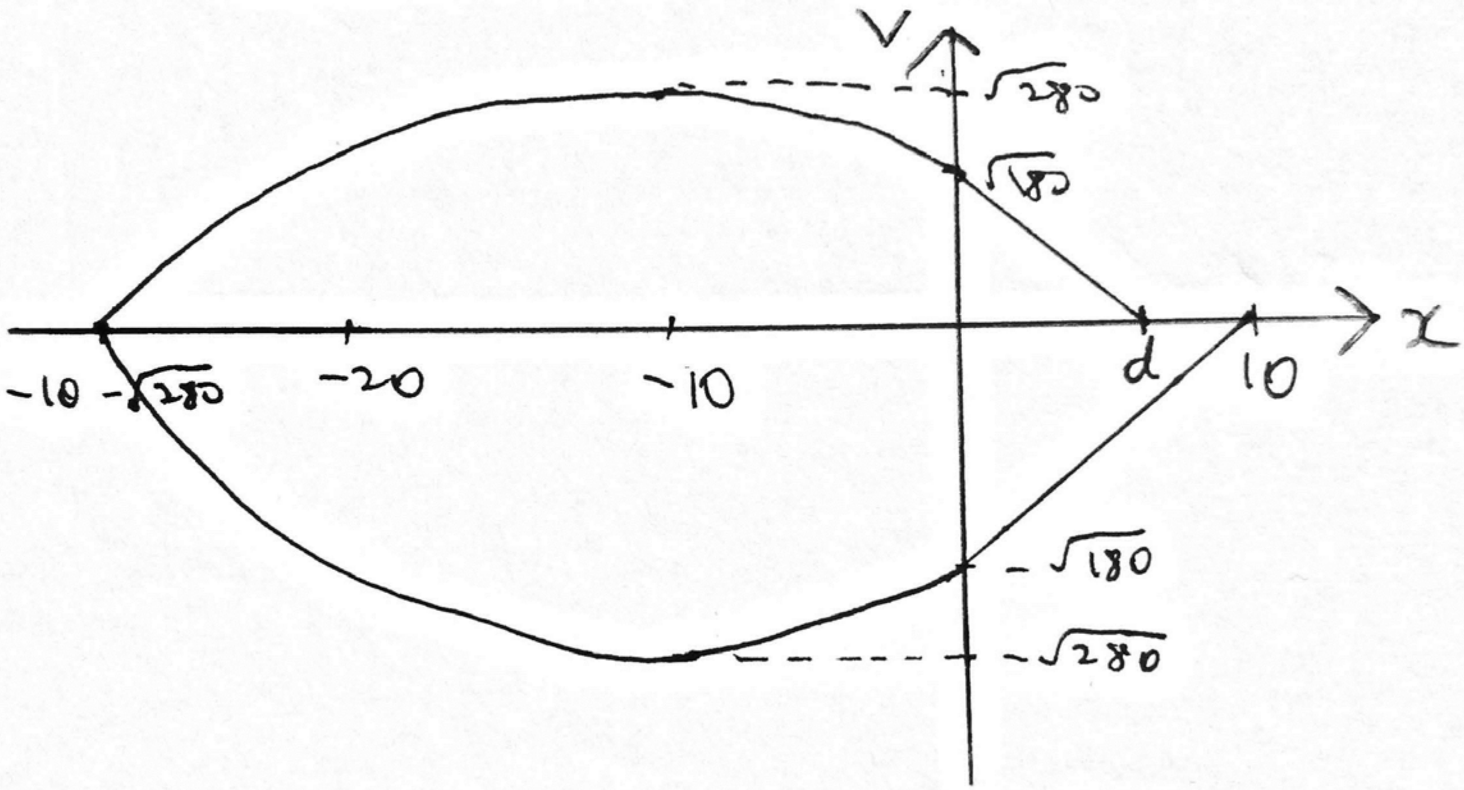
$t = 2s$





Now consider a case when, in addition, a friction force of a magnitude of 1 N is present for $x \geq 0$.

b1 Sketch how the velocity depends on x in that case. [6]



$$\text{Max KE} = (9 \times 10) + (10 \times 10 \times \tfrac{1}{2}) = 140\,J$$

$$\Rightarrow \text{max } v = \sqrt{280}$$

$$d = \frac{\frac{1}{2} \times (\sqrt{180})^2}{11} = 8.2\,m \ (2sf)$$

b2 How many meters this point like object travelled during the time when its position coordinate x was ≥ 0? [2]

Distance = 10 + 9

= 10 + 8.2

= 18.2 m