Centre for Education
in Mathematics and Computing

Euclid eWorkshop # 5

Solutions

SOLUTIONS

1. It is known that

$$\begin{aligned} \frac{t_{11}+t_{13}}{t_5+t_7} &= \frac{187500}{1500} \\ \frac{ar^{10}+ar^{12}}{ar^4+ar^6} &= 125 \\ \frac{ar^{10}(1+r^2)}{ar^4(1+r^2)} &= 125 \\ r^6 &= 125 \end{aligned}$$

Thus $r=\pm\sqrt{5}$. Hence $a=10$ and the first 3 terms are 10, $\pm 10\sqrt{5}$, 50.

2. Let d be the common difference in the arithmetic sequence. Then $b-c=-d$, $c-a=2d$ and $a-b=-d$. Thus we find the equations

$$\begin{aligned} -dx^2+2dx-d &= 0 \\ dx^2-2dx+d &= 0 \\ -d(x-1)^2 &= 0 \end{aligned}$$

and since $d \neq 0$ we have $x=1$.

3. We have $\dfrac{x+y}{2}=4$ and $xy=9$. Thus $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{x+y}{xy}=\dfrac{8}{9}$.

4. We let the numbers be $\dfrac{a}{r}$, a, ar. Thus $a^3=125$ and $a=5$, and the numbers are $\frac{5}{r},5,5r$. If the common difference in the arithmetic sequence is d, then

$$\begin{aligned} \frac{5-\frac{5}{r}}{5r-5} &= \frac{2d}{3d} \\ 3(5-\frac{5}{r}) &= 2(5r-5) \quad \text{dividing by } d \\ 3-\frac{3}{r} &= 2r-2 \quad \text{dividing by 5} \\ 0 &= 2r^2-5r+3 \\ 0 &= (2r-3)(r-1) \end{aligned}$$

So we have two solutions: $r=\frac{3}{2}$ gives the numbers $\dfrac{10}{3}$, 5, and $\dfrac{15}{2}$, while $r=1$ gives the (trivial) numbers $5,5,5$.

5. Our sum is

$$\begin{aligned}
\sum_{k=1}^{N} \frac{k^2+k}{2} &= \frac{\sum_{k=1}^{N} k^2 + \sum_{k=1}^{N} k}{2} \\
&= \frac{\frac{N(N+1)(2N+1)}{6} + \frac{N(N+1)}{2}}{2} \\
&= \frac{N(N+1)}{4}\left(\frac{2N+1}{3}+1\right) \\
&= \frac{N(N+1)(N+2)}{6} \\
\sum_{k=1}^{200} \frac{k^2+k}{2} &= \frac{200 \cdot 201 \cdot 202}{6} \\
&= 1353400.
\end{aligned}$$

6. Represent the angles with the variables $a-2d$, $a-d$, a, $a+d$ and $a+2d$. The sum of these is $540°$. Therefore $5a = 540°$ and $a = 108°$. So either $a-d = 90°$ or $a-2d = 90°$. So the largest angle is either $126°$ or $144°$.

7. We let the 4 positve integers be represented by k, kr, kr^2 and kr^3. Then

$$kr + kr^2 = 30 \tag{1}$$
$$k + kr^3 = 35 \tag{2}$$

Dividing (2) by (1) gives

$$\begin{aligned}
\frac{k+kr^3}{kr+kr^2} &= \frac{35}{30} \\
\frac{1+r^3}{r+r^2} &= \frac{7}{6} \text{ since } k \neq 0 \\
6r^3 - 7r^2 - 7r + 6 &= 0
\end{aligned}$$

By inspection we find that $r = -1$ is a solution. Using the factor theorem (see workshop 2), we arrive at

$$\begin{aligned}
(r+1)(2r-3)(3r-2) &= 0 \\
r &= -1, \frac{2}{3} \text{ or } \frac{3}{2}
\end{aligned}$$

Using $r = -1$, equation (2) gives $0k = 35$, which is impossible.
Using $r = \frac{2}{3}$ in (1), we find $k = 27$.
Using $r = \frac{3}{2}$ in (1) we find $k = 8$.
Both of these give the same list of numbers, and when arranged in increasing order they are $(a, b, c, d) = (8, 12, 18, 27)$.

8. The sequence is arithmetic if and only if $t_1 + t_3 = 2t_2$. There are 27 equally likely ways to pick 3 numbers, of which only 5 lead to such a sequence:
1,4,7
1,5,9
2,5,8
3,5,7
3,6,9
So the probability is $\frac{5}{27}$.

9. The average of the numbers, the middle number is $\frac{500}{25} = 20$. Thus the smallest is 8.

10. The difference is 2 so $n - 1 = \frac{1994-(-1994)}{2}$ and $n = 1995$.

11. (a) $S_1 = t_1 = 2$.
$S_2 = t_1 + t_2 = 8$ so $t_2 = 6$.
$S_3 = t_1 + t_2 + t_3 = 26$ so $t_3 = 18$.

(b)

$$\begin{aligned}\frac{t_{n+1}}{t_n} &= \frac{S_{n+1} - S_n}{S_n - S_{n-1}} \\ &= \frac{(3^{n+1} - 1) - (3^n - 1)}{(3^n - 1) - (3^{n-1} - 1)} \\ &= \frac{3^n \cdot 2}{3^{n-1} \cdot 2} \\ &= 3.\end{aligned}$$

12. The first such term is 42 and the last is 28000. So $n - 1 = \frac{28000-(42)}{7}$ and $n = 3995$.

13. We know $f(n+1) = f(n) + \frac{1}{3}$ so the sequence is arithmetic. So $f(100) = 2 + 99(\frac{1}{3}) = 35$.

14. Substituting for x and y, $-p + 2q = r$ so $q - p = r - q$ and we are done!

15. For any 3 term geometric sequence, $t_1 t_3 = (t_2)^2$. So

$$\begin{aligned}(a + 4d)(a + 15d) &= (a + 8d)^2 \\ a^2 + 19ad + 60d^2 &= a^2 + 16ad + 64d^2 \\ 3ad &= 4d^2 \\ d &= \frac{3}{4}a \text{ or } d = 0\end{aligned}$$

Thus the general term is

$$\begin{aligned}t_k &= a + (k-1)\frac{3}{4}a \\ &= \frac{a}{4}(3k + 1)\end{aligned}$$

We want to find terms $t_l t_m, t_n$ that form a geometric sequence, thus $t_l t_n = (t_m)^2$. We can do this by choosing integers a and b and letting
$3l + 1 = (2a)^2$
$3n + 1 = (2b)^2$ and
$3m + 1 = 4ab$.
Then $t_l = \frac{a}{4}(4a^2) = a^3$ and similarly, $t_n = ab^2$ and $t_m = a^2 b$. THus $t_l t_n = a^4 b^2 = (t_m)^2$. However we must add the extra condition that a and b must both be congruent to 1 modulo 3 or both be congruent to 2 modulo 3. There are infinitely many such pairs.

16. The sequence goes $5, 3, -2, -5, -3, 2, 5, 3, \ldots$. The sequence repeats in groups of 6 whose sum is 0. So the sum of 32 terms is $5 + 3 = 8$.

17. $t_4 = \frac{1}{3}t_2 = -\frac{1}{3}$

    $t_6 = \frac{3}{5}t_4 = -\frac{3}{5} \cdot \frac{1}{3}$

    $t_{1998} = -\frac{1995}{1997} \cdot \frac{1993}{1995} \cdot \frac{1991}{1993} \cdot \frac{1989}{1991} \cdot \ldots \ldots \cdot \frac{1}{3} = -\frac{1}{1997}$

18. Since the first term is 548 and the difference is -7 the sum is $S_n = \frac{n}{2}(1096 + (n-1)(-7))$. Thus the sum is negative when $(1096 + (n-1)(-7)) < 0$ and that is $n > 157$.