Centre for Education
in Mathematics and Computing

Euclid eWorkshop # 5

Sequences and Series

While the vast majority of Euclid questions in this topic area use formulae for arithmetic or geometric sequences, we will also include a few involving summations and different types of sequences.

TOOLKIT

Arithmetic Sequences

Description	Sequences with a common **difference**
General k^{th} term	$t_k = a + (k-1)d$ where a is the first term and d is this common difference
Sum of n terms	$S_n = \frac{n}{2}(a + t_n) = \frac{n}{2}(2a + (n-1)d)$
Spacing of terms	Because of the equal spacing of terms we have $t_k + t_l = t_m + t_n$ if and only if $k + l = m + n$

Geometric Sequences

Description	Sequences with a common **ratio**
General k^{th} term	$t_k = ar^{k-1}$ where a is the first term and r is the ratio
Sum of n terms	$S_n = \frac{a(1-r^n)}{(1-r)}$
Spacing of terms	Because of the equal spacing of terms we have $t_k t_l = t_m t_n$ if and only if $k + l = m + n$
Infinite sum	If the ratio r satisfies the condition $\lvert r \rvert < 1$, we can add an infinite number of terms using $S = \frac{a}{1-r}$

Other

Of course arithmetic and geometric sequences are a small subset of all sequences, even though they are emphasized in high school mathematics. Some extensions that frequently appear on contests often involve:

First n integers	$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
First n squares	$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
First n cubes	$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$
Telescoping series	If $t_k = u_k - u_{k-1}$ then $\sum_{k=1}^{n} t_k = \sum_{k=1}^{n}(u_k - u_{k-1}) = u_n - u_0$

SAMPLE PROBLEMS

1. What is the sum of all multiples of 7 or 11 less than 1000?

Solution

Since we are adding $(7 + 14 + 21 + 28 \ldots 994) + (11 + 22 + 33 + \ldots 990)$, we are adding two arithmetic sequences. However the multiples of 77 are included in both sequences and so must be subtracted (in order to avoid counting them twice) after we add the 2 sequences above. Therefore the required sum is

$$(7 + 14 + 21 + 28 \ldots 994) + (11 + 22 + 33 + \ldots 990) - (77 + 154 + \ldots 924).$$

Now since 994 is the 142^{nd} term in the first sequence we have the sum of the first sequence is $\frac{142}{2}(7 + 994)$. Thus the required sum is

$$\begin{aligned}
&\frac{142}{2}(7 + 994) + \frac{90}{2}(11 + 990) - \frac{12}{2}(77 + 924) \\
&= (71 + 45 - 6)(1001) \\
&= (110)(1001) \\
&= 110110
\end{aligned}$$

2. A sequence is given such that $t_1 = 1$ and $t_{n+1} = t_n + 3n^2 + 3n + 1$. Evaluate t_{100}.

Solution

Since the difference, $t_n - t_{n-1}$ is not constant, the series is not arithmetic. Now setting $n = 1$ we find $t_2 = 1 + 3 + 3 + 1 = 8$. Setting $n = 2$ we find $t_3 = 8 + 12 + 6 + 1 = 27$. These facts suggest $t_n = n^3$ for every n. To prove that $t_n = n^3$ is an alternate definition for the same sequence, we first note that $t_1 = 1 = 1^3$. Further, consider two adjacent terms in the sequence given by the alternate definition, i.e. $t_n = n^3$ and $t_{n+1} = (n+1)^3$. Then the difference between these terms is

$$\begin{aligned}
t_{n+1} - t_n &= (n + 1)^3 - (n)^3 \\
&= (n^3 + 3n^2 + 3n + 1) - n^3 \\
&= 3n^2 + 3n + 1 \\
t_{n+1} &= t_n + 3n^2 + 3n + 1
\end{aligned}$$

which matches the original definition of the sequence. We have proved that the original sequence can be expressed as $t_n = n^3$, and thus $t_{100} = 100^3$.

3. If a, b, $a+b$, and ab are positive numbers that form 4 consecutive terms in a geometric sequence, find a.

Solution

The ratios of successive terms will be equal since we have a geometric sequence. So

$$\frac{a}{b} = \frac{b}{a+b} = \frac{a+b}{ab} \quad (*)$$

Therefore,

$$\begin{aligned} a^2 + ab &= b^2 \\ b^2 - ab - a^2 &= 0 \\ \left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right) - 1 &= 0 \\ \left(\frac{b}{a}\right) &= \frac{1+\sqrt{5}}{2} \end{aligned}$$

where we have chosen the positive root since a and b are positive. Also from (*),

$$\begin{aligned} \frac{a}{b} &= \frac{a+b}{ab} \\ a^2 &= a+b \\ a &= 1 + \frac{b}{a} \\ &= \frac{3+\sqrt{5}}{2}. \end{aligned}$$

PROBLEM SET

1. In a geometric series, $t_5 + t_7 = 1500$ and $t_{11} + t_{13} = 187500$. Find all possible values for the first three terms.
2. Given that a, b and c are successive terms in an arithmetic sequence calculate x if $(b-c)x^2 + (c-a)x + (a-b) = 0$.
3. If x, 4, y are successive terms in an arithmetic sequence and x, 3, y are successive terms in a geometric sequence, calculate $\frac{1}{x} + \frac{1}{y}$.
4. Three different numbers, whose product is 125, are 3 consecutive terms in a geometric sequence. At the same time they are the first, third and sixth terms of an arithmetic sequence. Find the numbers.
5. The kth triangular number is given by $T_k = 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2} = \frac{k^2+k}{2}$. The first few triangular numbers are 1, 3, 6, 10, 15, 21. Find the sum of the first 200 triangular numbers.
6. If the interior angles of a pentagon form an arithmetic sequence and one interior angle is 90^o, find all possible values of the largest angle in the pentagon.
7. Find the 4 integers a, b, c and d that satisfy the following conditions:
   - the sum of b and c is 30
   - the sum of a and d is 35
   - the numbers $a < b < c < d$ are in geometric sequence
   - the sum of the squares of the 4 numbers is 1261
8. A sequence t_1, t_2, t_3 is formed by choosing t_1 at random from the set $\{1, 2, 3\}$, t_2 at random from the set $\{4, 5, 6\}$, and t_3 at random from the set $\{7, 8, 9\}$. What is the probability that t_1, t_2, t_3 is an arithmetic sequence?
9. The sum of 25 consecutive integers is 500. Determine the smallest of the 25 integers.
10. What is the number of terms in the arithmetic sequence -1994, -1992, -1990, . . ., 1992, 1994?
11. The sum of the first n terms of a sequence is $S_n = 3^n - 1$, where n is a positive integer.
    (a) If t_n represents the nth term of the sequence, determine t_1, t_2, t_3.
    (b) Prove that $\frac{t_{n+1}}{t_n}$ is constant for all values of n.
12. How many terms in the arithmetic sequence 7, 14, 21, . . . are between 40 and 28 001?
13. If f is a function such that $f(1) = 2$ and $f(n+1) = \frac{3f(n)+1}{3}$ for n = 1, 2, 3,. . ., what is the value of $f(100)$?
14. For the family of lines with equations of the form $px + qy = r$, and which all pass through the point $(-1, 2)$, prove that p, q, and r are consecutive terms of an arithmetic sequence.
15. An arithmetic sequence S has terms $t_1, t_2, t_3, \ldots$, where $t_1 = a$ and the common difference is d. The terms t_5, t_9, and t_{16} form a three-term geometric sequence with common ratio r. Prove that S contains an infinite number of three-term geometric sequences, all having the same common ratio r.
16. In the sequence $5, 3, -2, -5, ...,$, each term after the first two is constructed by taking the preceding term and subtracting the term before it. What is the sum of the first 32 terms in the sequence?

17. Consider the sequence $t_1 = 1, t_2 = -1$ and $t_n = \left(\frac{n-3}{n-1}\right) t_{n-2}$ where $n \geq 3$. What is the value of t_{1998}?

18. The *n*th term of an arithmetic sequence is given by $t_n = 555 - 7n$. If $S_n = t_1 + t_2 + \ldots + t_n$, determine the smallest value of *n* for which $S_n < 0$.