



**Centre for Education  
in Mathematics and Computing**

***Euclid eWorkshop # 1***  
***Logarithms and Exponents***



## TOOLKIT

If  $a$ ,  $b$ ,  $x$ , and  $y$  are real numbers and  $n$  is a nonzero integer then the rules for exponents are:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad a^0 = 1 \text{ if } a \neq 0 \quad a^{-x} = \frac{1}{a^x} \text{ if } a \neq 0$$

$$a^x a^y = a^{x+y} \quad \frac{a^x}{a^y} = a^{x-y} \text{ if } a \neq 0 \quad (a^x)^y = a^{xy}$$

$$a^x \cdot b^x = (ab)^x \quad \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x \text{ if } b \neq 0$$

Also,  $0^0$  is not defined if it is encountered using any of the above formulae.

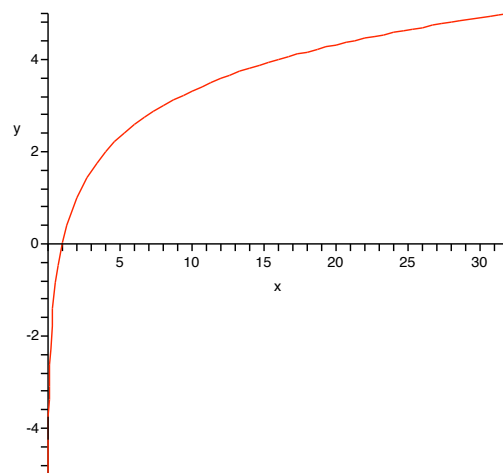
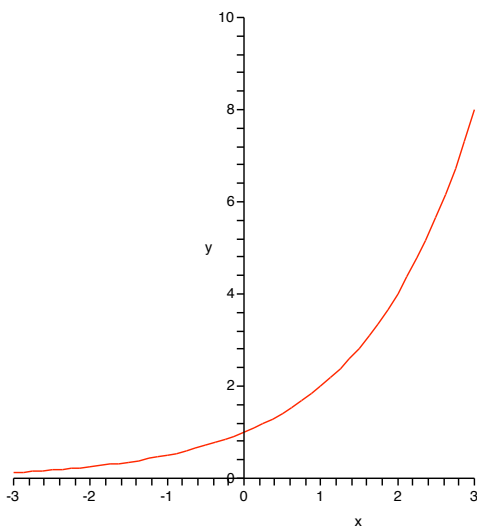
If  $a$ ,  $x$ , and  $y$  are nonzero real numbers then:

$$\log_a(xy) = \log_a x + \log_a y \quad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^y) = y \log_a x \quad \log_a(a^x) = a^{\log_a x} = x \quad \log_a 1 = 0$$

$$\log_a x = \frac{1}{\log_x a} \quad \frac{\log_a x}{\log_a y} = \log_y x$$

If  $f(x) = a^x$  then  $f^{-1}(x) = \log_a x$ . You should be able to sketch the graphs of both these functions. The graphs are shown for  $a = 2$  below.





## SAMPLE PROBLEMS

1. Calculate the ratio  $\frac{x}{y}$  if  $2 \log_5(x - 3y) = \log_5(2x) + \log_5(2y)$

### Solution

First we note that if the 3 logarithmic terms are to be defined in the original equation then their arguments must be positive. So  $x > 0$ ,  $y > 0$ , and  $x > 3y$ . Now

$$\begin{aligned} 2 \log_5(x - 3y) &= \log_5(2x) + \log_5(2y) \\ \log_5(x - 3y)^2 &= \log_5(4xy). \end{aligned}$$

Now since the log function takes on each value its range only once, this implies

$$\begin{aligned} (x - 3y)^2 &= 4xy \\ x^2 - 6xy + 9y^2 &= 4xy \\ x^2 - 10xy + 9y^2 &= 0 \\ (x - y)(x - 9y) &= 0 \end{aligned}$$

So  $\frac{x}{y} = 1$  or  $9$ . But  $\frac{x}{y} > 3$  from our restrictions so that  $\frac{x}{y} = 9$ .

2. If  $m$  and  $k$  are integers, find all solutions to the equation  $9(7^k + 7^{k+2}) = 5^{m+3} + 5^m$ .

### Solution

We factor both sides of this equation to arrive at:

$$\begin{aligned} 9(1 + 7^2)7^k &= 5^m(1 + 5^3) \\ 3^2 \cdot 2 \cdot 5^2 \cdot 7^k &= 5^m \cdot 2 \cdot 3^2 \cdot 7 \end{aligned}$$

Now since both sides of this equation are integers and have unique factorizations it follows that  $m = 2$  and  $k = 1$  is the only solution.

3. Determine the points of intersection of the curves  $y = \log_{10}(x - 2)$  and  $y = 1 - \log_{10}(x + 1)$ .

### Solution

Again the arguments of the log functions,  $x - 2$  and  $x + 1$  must be positive which implies that  $x > 2$ . Now



$$\begin{aligned}
 \log_{10}(x-2) &= 1 - \log_{10}(x+1) \\
 \log_{10}(x-2) + \log_{10}(x+1) &= 1 \\
 \log_{10}[(x-2)(x+1)] &= 1 \\
 (x-2)(x+1) &= 10 \\
 x^2 - x - 2 &= 10 \\
 x^2 - x - 12 &= 0 \\
 (x-4)(x+3) &= 0
 \end{aligned}$$

So  $x = 4$  or  $-3$ , but  $x > 2$  from our restrictions so  $x = 4$ . The point of intersection is  $(4, \log_{10} 2)$  or  $(4, 1 - \log_{10} 5)$ . Since  $\log_{10} 2 + \log_{10} 5 = 1$ , these are equivalent answers.

4. Solve for  $x$  if  $\log_2(9 - 2^x) = 3 - x$ .

### Solution

Once again the argument of the log must be positive, implying that  $9 > 2^x$ .

$$\begin{aligned}
 \log_2(9 - 2^x) &= 3 - x \\
 (9 - 2^x) &= 2^{3-x} = \frac{8}{2^x}
 \end{aligned}$$

Substituting  $y = 2^x$  we have:

$$\begin{aligned}
 9 - y &= \frac{8}{y} \\
 y^2 - 9y + 8 &= 0
 \end{aligned}$$

Thus,  $y = 1$  or  $y = 8$ . Substituting back in, we see that  $x = 0$  or  $x = 3$ . Both of these satisfy the restriction.

5. The graph of  $y = m^x$  passes through the points  $(2, 5)$  and  $(5, n)$ . What is the value of  $mn$ ?

### Solution

From the given information,  $m^2 = 5$  and  $n = m^5$ . Thus

$$\begin{aligned}
 m &= \pm\sqrt{5} \\
 n &= (\pm\sqrt{5})^5 \\
 mn &= (\sqrt{5})^6 = 125.
 \end{aligned}$$



## PROBLEM SET

1. Determine  $x$  such that  $\log_x 2 + \log_x 4 + \log_x 8 = 1$ .

2. Determine the values of  $x$  such that  $12^{2x+1} = 2^{3x+7} \cdot 3^{3x-4}$ .

3. What is the sum of the following series?

$$\log_{10} \frac{3}{2} + \log_{10} \frac{4}{3} + \log_{10} \frac{5}{4} + \cdots \cdots \log_{10} \frac{200}{199}.$$

4. If  $x^3 y^5 = 2^{11} \cdot 3^{13}$  and  $\frac{x}{y^2} = \frac{1}{27}$ , solve for  $x$  and  $y$ .

5. If  $\log_8 3 = k$  then express  $\log_8 18$  in terms of  $k$ .

6. Solve the equations for the point of intersection of the graphs of  $y = \log_2(2x)$  and  $y = \log_4 x$ .

7. The points  $A(x_1, y_1)$  and  $B(x_2, y_2)$  lie on the graph of  $y = \log_a x$ . Through the midpoint of  $AB$  a horizontal line is drawn to cut the curve at  $C(x_3, y_3)$ . Show that  $(x_3)^2 = x_1 x_2$ .

8. The function  $y = ax^r$  passes through the points  $(2, 1)$  and  $(32, 4)$ . Calculate the value of  $r$ .

9. If  $2^{x+3} + 2^x = 3^{y+2} - 3^y$  and  $x$  and  $y$  are integers, determine the values of  $x$  and  $y$ .

10. If  $f(x) = 2^{4x-2}$ , calculate, in simplest form,  $f(x) \cdot f(1-x)$ .

11. Find all values of  $x$  such that  $\log_5(x-2) + \log_5(x-6) = 2$ .

12. Prove that  $a$ ,  $b$ , and  $c$  are 3 numbers that form a geometric sequence if and only if  $\log_x a$ ,  $\log_x b$  and  $\log_x c$  form an arithmetic sequence.