Centre for Education
in Mathematics and Computing

Euclid eWorkshop # 2

Functions and Equations

TOOLKIT

Parabolas

The quadratic $f(x) = ax^2 + bx + c$ (with a,b,c real and $a \neq 0$) has two zeroes given by $r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
These roots are :

- real and distinct if the discriminant $\Delta = b^2 - 4ac > 0$
- real and equal if the discriminant $\Delta = b^2 - 4ac = 0$
- distinct and non-real if the discriminant $\Delta = b^2 - 4ac < 0$

The sum of these roots is $r_1 + r_2 = \dfrac{-b}{a}$ and their product $r_1 r_2 = \dfrac{c}{a}$.

Since $y = ax^2 + bx + c = a\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a}$, the vertex of the graph is located at $\left(-\dfrac{b}{2a}, \dfrac{4ac - b^2}{4a}\right)$.
You should be able to sketch the six generic cases of the graph of the parabola that occurs when $a > 0$ or < 0 and $\Delta > 0, < 0$, or $= 0$.

Polynomials

Remainder Theorem and Factor Theorem
The Remainder Theorem states that when a polynomial $p(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots a_n$, of degree n, is divided by $(x - k)$ the remainder is $p(k)$. The factor theorem then follows: $p(k) = 0$ if and only if $(x - k)$ is a factor of $p(x)$. A polynomial equation of degree n has at most n real roots.

Rational Root Theorem
The rational root theorem states that all rational roots $\frac{p}{q}$ have the property that p and q are factors of the last and first coefficient, a_n and a_0 respectively.

Function Transformations
The graph of $y = p(x)$ or $y = f(x)$ can be used to graph its various transformed cousins:

$y = p(x) + k$ is shifted up k units; $(k > 0)$

$y = p(x - k)$ is shifted right k units; $(k > 0)$

$y = kp(x)$ is stretched vertically by a factor of k; $(k > 0)$

$y = p(\frac{x}{k})$ is stretched horizontally by a factor of k; $(k > 0)$

$y = -p(x)$ is reflected in the x axis;

$y = p(-x)$ is reflected in the y axis;

$x = f(y)$ or $y = f^{-1}(x)$ is reflected across the line $y = x$.

SAMPLE PROBLEMS

1. If $x^2 - x - 2 = 0$, determine all possible values of $1 - \frac{1}{x} - \frac{6}{x^2}$.

Solution

$$\text{We have} \quad 1 - \frac{1}{x} - \frac{6}{x^2} = \frac{x^2 - x - 6}{x^2} = \frac{x^2 - x - 2 - 4}{x^2} = \frac{-4}{x^2}$$

$$\begin{aligned} \text{Since} \quad x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ \text{Thus} \quad x &= 2 \text{ or } -1 \end{aligned}$$

Therefore possible values are -1 and -4.

2. If the graph of the parabola $y = x^2$ is translated to a position such that its x intercepts are $-d$ and e and its y intercept is $-f$, (where d,e,$f > 0$), show that $de = f$.

Solution 1 (easy)

Since the x intercepts are $-d$ and e the parabola must be of the form $y = a(x + d)(x - e)$. Also since we have only translated $y = x^2$ it follows that $a = 1$. Now setting $x = 0$ we have $-f = -de$ and the results follows.

Solution 2 (harder)

Let the parabola be $y = ax^2+bx+c$. Now, as in the first solution, $a = 1$. Then solving for the x and y intercepts we find $e = \frac{-b + \sqrt{b^2 - 4c}}{2}$, $-d = \frac{-b - \sqrt{b^2 - 4c}}{2}$ and$-f = c$. Now straight forward multiplication gives $-de = \frac{-b - \sqrt{b^2 - 4c}}{2} \cdot \frac{-b + \sqrt{b^2 - 4c}}{2} = \frac{b^2 - b^2 + 4c}{4} = c = -f$ as required!

3. Find all values of x such that $x + \frac{36}{x} \geq 13$.

Solution

First we note that $x \neq 0$. If $x > 0$, we can multiply the equation by this positive quantity and arrive at $x^2 - 13x + 36 \geq 0$ or $(x - 4)(x - 9) \geq 0$. Since $x > 0$ this gives $4 \geq x > 0$ or $x \geq 9$. If $x < 0$ the left side of the inequality is negative, which means it is not greater than 13. Therefore $0 < x \leq 4$ or $x \geq 9$.

4. If a polynomial leaves a remainder of 5 when divided by $x - 3$ and a remainder of -7 when divided by $x + 1$, what is the remainder when the polynomial is divided by $x^2 - 2x - 3$?

Solution

We observe that when we divide by a second degree polynomial the remainder will generally be linear. Thus the division statement becomes
$p(x) = (x^2 - 2x - 3)q(x) + ax + b$ (*)
where $p(x)$ is the polynomial, $q(x)$ is the quotient polynomial and $ax + b$ is the remainder. Now we observe that the remainder theorem states $p(3) = 5$ and $p(-1) = -7$. Also we notice that $x^2 - 2x - 3 = (x - 3)(x + 1)$. Thus substituting $x = 3$ and -1 into (*) we have:

$$p(3) = 5 = 3a + b$$
$$p(-1) = -7 = -a + b$$

Solving these equations $a = 3$ and $b = -4$; the remainder is $3x - 4$.

PROBLEM SET

1. If x and y are real numbers, determine all solutions (x, y) of the system of equations
$$x^2 - xy + 8 = 0$$
$$x^2 - 8x + y = 0$$

2. The parabola defined by the equation $y = (x - 1)^2 - 4$ intersects the x-axis at points P and Q. If (a,b) is the midpoint of PQ, what is the value of a?

3. (a) The equation $y = x^2 + 2ax + a$ represents a parabola for all real values of a. Prove that there exists a common point through which all of these parabolas pass, and determine the coordinates of this point.
   (b) The vertices of these parabolas lie on a curve. Prove that this curve is itself a parabola whose vertex is the common point found in part (a).

4. (a) Sketch the graph of the equation $y = x(x - 4)^2$. Label all intercepts.
   (b) Solve the inequality $x(x - 4)^2 \geq 0$.

5. Determine all real values of p and r that satisfy the following system of equations:
$$p + pr + pr^2 = 26$$
$$p^2r + p^2r^2 + p^2r^3 = 156$$

6. A quadratic equation $ax^2 + bx + c = 0$ (where a, b, and c are not zero), has real roots. Prove that a, b, and c cannot be consecutive terms in a geometric sequence.

7. A quadratic equation $ax^2 + bx + c = 0$ (where x, a, b, and c are integers and $a \neq 0$), has integer roots. If a, b, and c are consecutive terms in an arithmetic sequence, solve for the roots of the equation.

8. Solve this equation for x:
$$(x^2 - 3x + 1)^2 - 3(x^2 - 3x + 1) + 1 = x.$$

9. The parabola $y = (x - 2)^2 - 16$ has its vertex at point A and its larger x intercept at point B. Find the equation of the line through A and B.

10. Solve the equation $(x - b)(x - c) = (a - b)(a - c)$ for x.

11. Given that $x = -2$ is a solution of $x^3 - 7x - 6 = 0$, find the other solutions.

12. Find the value of a such that the equation below in x has real roots, the sum of whose squares is a minimum.
$$4x^2 + 4(a - 2)x - 8a^2 + 14a + 31 = 0.$$

13. If $f(x) = \dfrac{3x - 7}{x + 1}$ and $g(x)$ is the inverse of $f(x)$, then determine the value of $g(2)$.

14. If $(-2,7)$ is the maximum point for the function $y = -2x^2 - 4ax + k$, determine k.

15. The roots of $x^2 + cx + d = 0$ are a and b and the roots of $x^2 + ax + b = 0$ are c and d. If a, b, c and d are nonzero, calculate $a + b + c + d$.

16. If $y = x^2 - 2x - 3$ then determine the minimum value of $\dfrac{y - 4}{(x - 4)^2}$.