Centre for Education
in Mathematics and Computing

Euclid eWorkshop # 6

Solutions

SOLUTIONS

1.

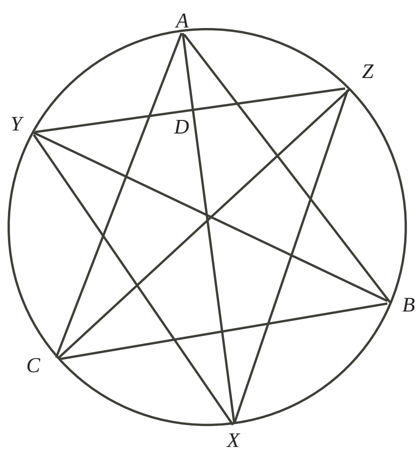
$$\begin{aligned}\angle APB &= \angle ACB + \angle CBD \quad \text{(exterior angle of } \triangle BPC\text{)}\\ &= \frac{1}{2}\angle AOB + \frac{1}{2}\angle COD.\end{aligned}$$

2. Let X be the point of intersection of PR and QS. From question 1,

$$\begin{aligned}\angle PXQ &= \frac{1}{2}(\angle POQ + \angle ROS)\\ &= \frac{1}{2}(\angle POB + \angle BOQ + \angle ROD + \angle DOS)\\ &= \frac{1}{4}(\angle AOB + \angle BOC + \angle COD + \angle DOA)\\ &= 90^\circ.\end{aligned}$$

3. The two triangles can be used to find two opposite interior angles in the cyclic quadrilateral, which are $180^\circ - 5x$ and $180^\circ - 4x$. Since these add to 180°, we have $x = 20^\circ$.

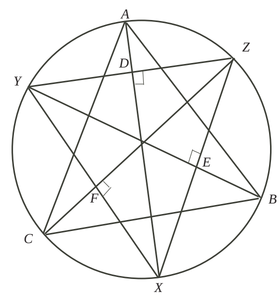
4. We prove first that the bisectors of the angles of $\triangle ABC$ are altitudes of $\triangle XYZ$. Let D be the point of intersection of AX and YZ.



$$\begin{aligned}\text{We know } \angle ACZ + \angle YBC + \angle CAX &= \frac{180^\circ}{2} \quad \text{since these angles are half the angles of } \triangle ABC\\ \angle DXZ + \angle YZC + \angle CZX &= 90^\circ \quad \text{subtended by chords } AZ, YC, CX \text{ respectively}\\ \angle DXZ + \angle DZX &= 90^\circ.\end{aligned}$$

Thus in $\triangle DZX$, $\angle ZDX = 90^\circ$. So XA is an altitude of $\triangle XYZ$. A similar argument shows BY and CZ are also altitudes.

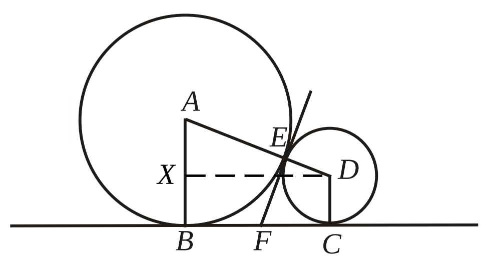
We now prove that the altitudes of $\triangle XYZ$ bisect the angles of $\triangle ABC$.
We have with $AX \perp YZ$ at D, $BY \perp XZ$ at E, $CZ \perp XY$ at F.



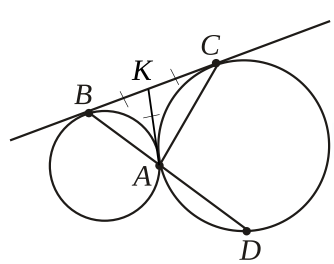
$$\begin{aligned}\text{Now } \angle BAX &= \angle BYX \quad \text{(common chord } BX\text{)}\\ &= 90^\circ - \angle ZXY \quad (\triangle EXY)\\ &= 90^\circ - \angle ZXF \quad \text{(relabel)}\\ &= \angle XZC \quad (\triangle XZF)\\ &= \angle XAC \quad \text{(common chord } XC\text{)}\end{aligned}$$

We have shown that chord AX bisects $\angle BAC$. A similar argument can be used to shown that chords BY and CZ bisect $\angle ABC$ and $\angle ACB$ respectively.

5. Draw the perpendicular DX from D to AB. The triangle ADX has sides $AD = 4 + 9 = 13$ and $XA = 9 - 4 = 5$. Then $DX = 12$ using the theorem of Pythagoras. But $DCBX$ is a rectangle so $BC = 12$. But $FB = FE = FC$ since the 3 segments are tangents from point F. Thus $FE = 6$.



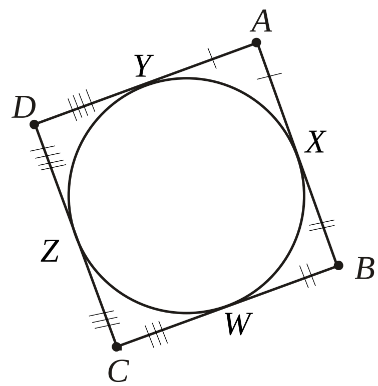
6. (a) Draw the tangent at A and let this tangent intersect BC at K.



Then as in question 5, $KB = KA = KC$. Thus a circle drawn on BC as diameter passes through A and $\angle BAC = 90^\circ$.

(b) By (a), thus $\angle CAD = 90^\circ$ and CD is a diameter.

7. Label the points of tangency X, Y, Z, and W as shown in the diagram.



Then $AX = AY$, $BX = BW$, $CW = CZ$ and $DZ = DY$ since all pairs of tangents from an external point are equal. So

$$\begin{aligned} AD + BC &= AY + YD + BW + WC \\ &= AX + DZ + BX + CZ \\ &= AB + CD. \end{aligned}$$

8. Draw the line KAL to be tangent at A. Label the points X and Y where AB and AC intersect the inner circle. We use the Tangent Chord Theorem many times to obtain:

$$\begin{aligned} \angle KAB = \angle KAX &= \angle AYX \quad \text{(TCT in small circle)} \\ &= \angle ADX \quad \text{(common chord } AX\text{)} \end{aligned}$$

$$\begin{aligned} \angle LAY &= \angle AXY \quad \text{(TCT in small circle)} \\ &= \angle ABC \quad \text{(TCT in big circle)} \\ &= \angle ABD \end{aligned}$$

$$\angle DAX = \angle XDB \quad \text{(TCT small circle)}$$

$$\angle DAY = \angle DXY \quad \text{(common chord } DY\text{)}$$

In $\triangle BAD$ we have

$$\angle BAD + \angle ABD + \angle ADX + \angle XDB = 180^\circ.$$

Replacing equal angles from above gives

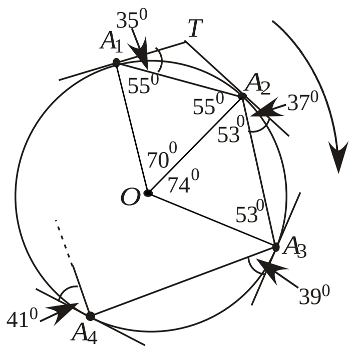
$$\angle BAD + \angle LAY + \angle KAB + \angle DAX = 180^\circ.$$

But since KAL is a straight line we have

$$\angle BAD + \angle LAY + \angle KAB + \angle YAD = 180^\circ.$$

Thus $\angle DAX = \angle YAD$ and AD bisects $\angle BAC$.

9. Extend the tangents at A_1 and A_2 to meet at point T and join A_1 and A_2 to the centre O. Since tangents from an external point are equal, $\triangle TA_1A_2$ is isosceles and $\angle A_1A_2T = \angle A_2A_1T = 35°$. We now consider $\triangle OA_1A_2$. Since the radii are perpendicular to the tangents, we have $\angle OA_1A_2 = \angle OA_2A_1 = 55°$ and thus $\angle A_1OA_2 = 70°$.



Using the same procedure, we get $\angle A_2OA_3 = 74°$ and in general,

$$\angle A_iOA_{i+1} = [70 + 4(i-1)]°.$$

(For convenience, we will omit the ° sign; all angles are understood to be in degrees.)
The particle will return to A_1 for the first time when $\angle A_1OA_2 + \angle A_2OA_3 + \angle A_3OA_4 + ... + \angle A_nOA_{n+1}$ is an integer multiple of 360° so that $A_{n+1} = A_1$. But

$$\begin{aligned}
&\angle A_1OA_2 + \angle A_2OA_3 + \angle A_3OA_4 + ... + \angle A_nOA_{n+1}\\
&= 70 + 74 + 78 + ... + [70 + 4(n-1)]\\
&= 70n + 4\frac{(n-1)(n)}{2}\\
&= 70n + 2n^2 - 2n\\
&= 2n^2 + 68n
\end{aligned}$$

So we want a small integer k such that

$$\begin{aligned}
\angle A_1OA_2 + \angle A_2OA_3 + \angle A_3OA_4 + ... + \angle A_nOA_{n+1} &= 360k\\
2n^2 + 68n &= 360k\\
n^2 + 34n &= 180k\\
n^2 + 34n + 289 &= 180k + 289 \quad \text{(completing the square)}\\
(n+17)^2 &= 180k + 289
\end{aligned}$$

Since the left side is a perfect square, the right side must be as well. Trying $k = 1, 2, 3...$, we find that $k = 6$ is the smallest whole value for which the right side is a perfect square. When $k = 6$, we have $n = 20$.

We can verify that $70 \cdot 20 + 4\frac{19 \cdot 20}{2} = 2160$, which is an integer multiple of 360.