Centre for Education
in Mathematics and Computing

Euclid eWorkshop # 6

Circle Geometry

CIRCLE GEOMETRY

Geometry and more specifically the geometry of the circle represents an area of mathematics from which relatively difficult and interesting problems for the Euclid contest are frequently taken. Although this is a very broad content area, we present only a brief outline of some of the more elementary results of the geometry of the circle. We will assume that the student has some knowledge of the elementary properties of triangles, including congruence and similarity, as well as the properties of parallelograms, rhombi and trapezoids that follow from these properties of triangles.

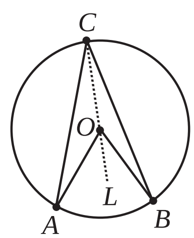
The 'Star Trek' Theorem

The central angle subtended by any arc is twice any of the inscribed angles on that arc. In other words, in the accompanying figure $\angle AOB = 2\angle ACB$.

Demonstration

This theorem is the lynchpin of all the results that follow.

It is useful to give a brief demonstration of this property to see that if follows directly from the properties of isosceles triangles.



In the diagram we draw the line connecting *C* and *O* and let a point on *CO* extended be *L*. Since *OA*, *OB* and *OC* are radii we have that the triangles *OAC* and *OBC* are isosceles. Thus $\angle OAC = \angle OCA$. Furthermore $\angle OAC + \angle OCA = \angle AOL$, the exterior angle of the triangle. Therefore $\angle OCA = \frac{1}{2}(\angle AOL)$. Similarly $\angle OCB = \frac{1}{2}(\angle BOL)$.

Adding these 2 results $\angle ACB = \angle ACO + \angle OCB = \frac{1}{2}(\angle AOL + \angle LOB) = \frac{1}{2}\angle AOB$.

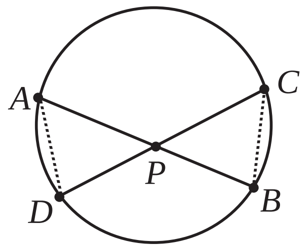
Extensions

See if you can extend this theorem to obtain each of the following:

1. Show that if the chord AB is a diameter then $\angle ACB = 90°$. (The angle subtended by a diameter is a right angle).
2. Show that the result is still true if $\angle AOB$ is greater than $180°$.
3. Show that the result is true in the case where the point C is chosen so that the segments AC and OB intersect, that is, when C is moved along the circle towards the point B. (This proof will use differences where the earlier proof uses sums).
4. Show that if C_1 and C_2 are 2 different choices for the position of the point C along the arc AB then $\angle AC_1B = \angle AC_2B$. This result is described as 'angles subtended by the same arc(or chord) are equal'. (The two points must both lie on the same side of the chord).
5. If C_1 and C_2 are two points on the circle, one on the minor arc AB and the other on the major arc AB, prove that $\angle AC_1B + \angle AC_2B = 180°$ (the opposite angles of a cyclic quadrilateral are supplementary).

The 'Crossed Chord' Theorem

If two chords AB and CD of a circle intersect at the point P as shown in the diagram, prove that $(PA)(PB) = (PC)(PD)$.

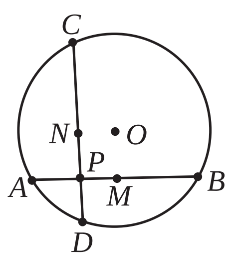


Proof

Draw the line segments joining the points AD and BC. Then we observe using the result (4) above that $\angle BCD = \angle BAD$ and $\angle ADC = \angle ABC$. Thus ΔADP is similar to ΔCBP by the angle-angle similarity. Thus $\dfrac{PA}{PC} = \dfrac{PD}{PB}$, and by cross-multiplying we arrive at $(PA)(PB) = (PC)(PD)$.

Problem

Two perpendicular chords AB and CD intersect at P. If $PA = 4$, $PB = 10$ and $CD = 13$, calculate the length of the radius of the circle.

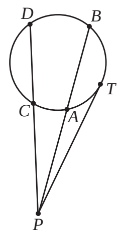


Solution

We let x represent the length PC. Then, using the **Crossed Chord Theorem**, $x(13 - x) = 4(10)$ and $x = 5$ or 8. Now we label the midpoints AB and CD as M and N. Thus $MB = \dfrac{10+4}{2} = 7$ and $NC = \dfrac{8+5}{2} = \dfrac{13}{2}$. Thus $NP = \left|8 - \dfrac{13}{2}\right|$ or $\left|5 - \dfrac{13}{2}\right| = \dfrac{3}{2}$. However the line segment joining the centre of a circle to the mid-point of a chord is perpendicular to the chord and the chords were given perpendicular, so $OMPN$ is a rectangle. Thus $OM = NP = \dfrac{3}{2}$ and we can use the theorem of Pythagoras in triangle OMB to calculate the radius $r = OB = \sqrt{7^2 + \left(\dfrac{3}{2}\right)^2} = \dfrac{\sqrt{205}}{2}$.

Extension

In the diagram, PAB and PCD are two secants of the same circle and they intersect at the point P outside the circle. Prove that $(PA)(PB) = (PC)(PD)$.



Solution

Consider $\triangle PAD$ and $\triangle PCB$. Since both share $\angle APC$ we need only have one more pair of angles equal to establish similarity. But $\angle ADC$ and $\angle ABC$ both are subtended by the arc *AC* and so are equal. Thus ΔPAD is similar to ΔPCB and using the same logic as above we arrive at $(PA)(PB) = (PC)(PD)$.

Now imagine a series of secants passing through P, each intersecting the circle at two points but with the chord that is within the circle getting shorter and shorter as the secant approaches the point where it intersects the circle at a single point. In this case the secant becomes a tangent to the circle at the limiting point of intersection which we label T. We notice that as this process takes place, PA approaches PT and PB approaches PT. Thus we have $(PA)(PB) = (PC)(PD) = (PT)^2$. Try to prove this last statement directly using similarity and result (IV) below!

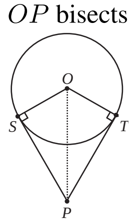
Other Important Properties of Tangents

If P is a point outside of a circle and we draw the two tangents to the circle PT and PS then the following results follow:

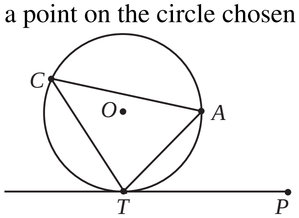
I. A tangent at a point on a circle is perpendicular to the radius drawn to the point. (OT is perpendicular to PT)

II. $PS = PT$: Tangents to a circle from an external point are equal.

III. OP bisects the angle between the tangents. ($\angle TPS$).



IV. **Tangent Chord Theorem:** Given that TA is any chord of a circle and PT is tangent to the circle at T. If C is a point on the circle chosen to be on the side of the chord opposite to the tangent then $\angle TCA = \angle PTA$.

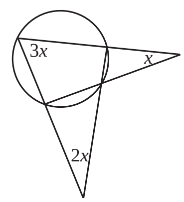


Proof

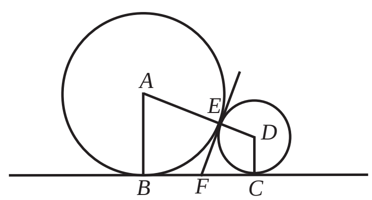
Since we know OT is perpendicular to TP we draw in the radii OT and OA. Since the tangent is perpendicular to the radius, $\angle PTA = 90° - \angle ATO = \frac{1}{2}(180° - \angle ATO - \angle OAT) = \frac{1}{2}(\angle AOT) = \angle ACT$.

PROBLEM SET

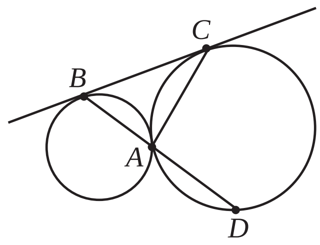
1. In a circle with centre O, two chords AC and BD intersect at P. Show that $\angle APB = \frac{1}{2}(\angle AOB + \angle COD)$.
2. If the points A, B, C and D are any 4 points on a circle and P, Q, R and S are the midpoints of the arcs AB, BC, CD and DA respectively, show that PR is perpendicular to QS.
3. Calculate the value of x.



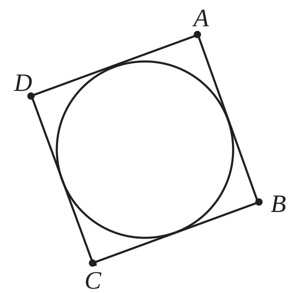
4. The three vertices of triangle ABC lie on a circle. Chords AX, BY, CZ are drawn within the interior angles A, B, C of the triangle. Show that the chords AX, BY and CZ are the altitudes of triangle XYZ if and only if they are the angle bisectors of triangle ABC.
5. As shown in the diagram, a circle with centre A and radius 9 is tangent to a smaller circle with centre D and radius 4. Common tangents EF and BC are drawn to the circles making points of contact at E, B and C. Determine the length of EF.



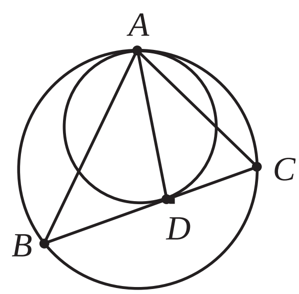
6. In the diagram, two circles are tangent at A and have a common tangent touching them at B and C respectively.
   (a) Show that $\angle BAC = 90°$. (Hint: with touching circles it is usual to draw the common tangent at the point of contact!)
   (b) If BA is extended to meet the second circle at D show that CD is a diameter.



7. If $ABCD$ is a quadrilateral with an inscribed circle as shown, prove that $AB + CD = AD + BC$.



8. In this diagram, the two circles are tangent at A. The line BDC is tangent to the smaller circle. Show that AD bisects $\angle BAC$.



9. Starting at point A_1 on a circle, a particle moves to A_2 on the circle along chord A_1A_2 which makes a clockwise angle of $35°$ to the tangent to the circle at A_1. From A_2 the particle moves to A_3 along chord A_2A_3 which makes a clockwise angle of $37°$ to the tangent at A_2. The particle continues in this way. From A_k it moves to A_{k+1} along chord A_kA_{k+1} which makes a clockwise angle of $(33 + 2k)°$ to the tangent to the circle at A_k. After several trips around the circle, the particle returns to A_1 for the first time along chord A_nA_1. Find the value of n.