

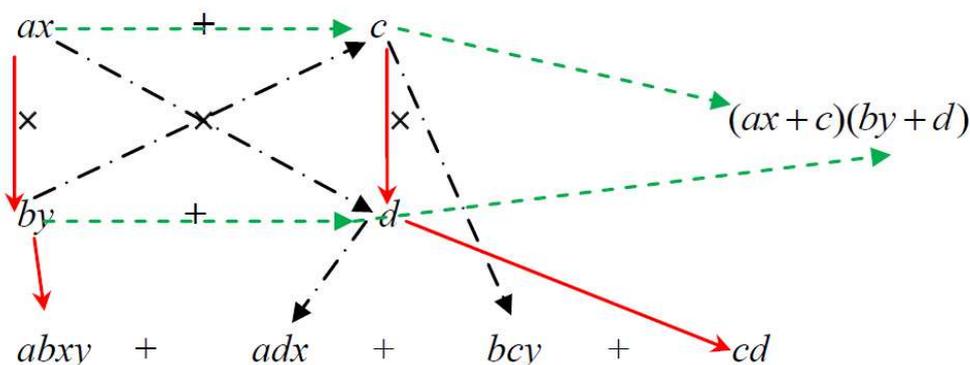
## FACTOR BY CROSS MULTIPLICATIONS

### Theorem:

$$abxy + adx + bcy + cd = (ax + c)(by + d)$$

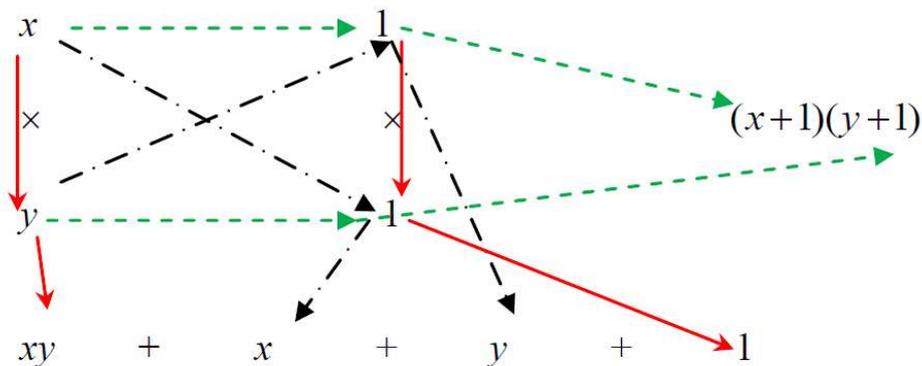
### Procedures:

- Step 1: Vertical multiplication:  $(ax) \times (by)$
- Step 2: Cross multiplication:  $(ax) \times (d)$
- Step 3: Cross multiplication:  $(by) \times (c)$
- Step 4: Vertical multiplication:  $(c) \times (d)$
- Step 5: Horizontal addition:  $(ax + c)$
- Step 6: Horizontal addition:  $(by + d)$
- Step 7: Making the product:  $(ax + c) \times (by + d)$ .

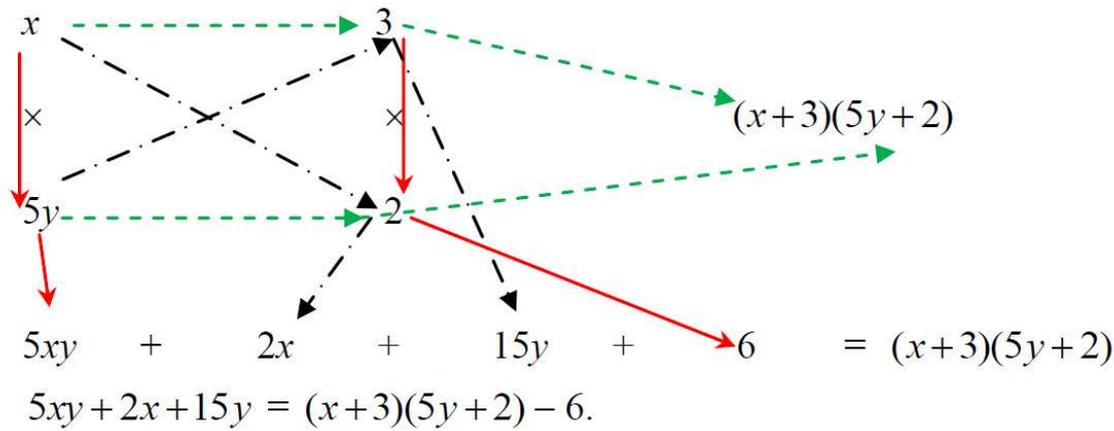


### Illustrations

(1)  $xy + x + y + 1$



(2)  $5xy + 2x + 15y$



**One more formula**

$xyz + xy + yz + zx + x + y + z + 1 = (x+1)(y+1)(z+1)$

**FACTORING USING FORMULAS**

Below is a list of useful formulas for factoring.

**Differences of two squares:**

$a^2 - b^2 = (a+b)(a-b)$

Note:

- (1)  $a - b$  and  $a + b$  have the same parity.
- (2) For positive integers  $a$  and  $b$ ,  $a + b > a - b$ .
- (3) If  $a^2 - b^2 = (a+b)(a-b) = n$ , where  $n$  is even, then  $n$  must be divisible by 4.

**Square of differences and sum of two variables:**

$a^2 - 2ab + b^2 = (a-b)^2$

$a^2 + 2ab + b^2 = (a+b)^2$

**Square of sum of three variables:**

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a + b + c)^2$$

**Differences and sum of two cubes:**

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

**Cube of differences and sum of two variables:**

$$a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$$

$$a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$$

**One more formula**

$$a^3 + b^3 + c^3 - 3acb = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

**EXAMPLES**

**Example 1.** What is the product of positive integers  $a$  and  $b$  such that  $a > b$  and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = 1?$$

- (A) 1            (B) 2            (C) 3            (D) 4            (E) 6

**Solution:** (E).

The given equation can be written as  $b + a + 1 = ab$  or

$$ab - a - b = 1 \quad \Rightarrow \quad (a - 1)(b - 1) = 2.$$

We have  $(a - 1) = 2$  and  $(b - 1) = 1$ .

So  $a = 3$  and  $b = 2$ . The product is 6.

**Example 2.** (1993 AMC 12) How many ordered pairs  $(m, n)$  of positive integers

are solutions to  $\frac{4}{m} + \frac{2}{n} = 1$ ?

- (A) 1            (B) 2            (C) 3            (D) 4            (E) more than 4

**Solution:** (D).

Method 1 (official solution):

Since  $m$  and  $n$  must both be positive, it follows that  $n > 2$  and  $m > 4$ . Because

$$\frac{4}{m} + \frac{2}{n} = 1 \Rightarrow (m-4)(n-2) = 8,$$

we need to find all ways of writing 8 as a product of positive integers.

The four ways,  $1 \cdot 8$ ,  $2 \cdot 4$ ,  $4 \cdot 2$  and  $8 \cdot 1$ , correspond to the four solutions  $(m, n) = (5, 10)$ ,  $(6, 6)$ ,  $(8, 4)$  and  $(12, 3)$ .

Method 2 (our solution):

$$\begin{aligned} \frac{4}{m} + \frac{2}{n} = 1 &\Rightarrow \frac{4n+2m}{mn} = 1 \Rightarrow 4n+2m = mn \Rightarrow m(n-2) = 4n \\ \Rightarrow m = \frac{4n}{n-2} &= \frac{4(n-2)+8}{n-2} = 4 + \frac{8}{n-2}. \end{aligned}$$

Since  $m$  is a positive integer,  $n-2$  must be a factor of 8.

So  $n = 3, 4, 6$ , and  $10$ .  $(m, n) = (5, 10)$ ,  $(6, 6)$ ,  $(8, 4)$  and  $(12, 3)$ .

**Example 3.** The product of three prime numbers is five times the sum of these prime numbers and the product is also divisible by 5. Find the largest of these prime numbers.

- (A) 2            (B) 3            (C) 5            (D) 7            (E) 11

**Solution:** (D).

Since the product is divisible by 5, one of these prime numbers must be 5.

Let  $p$  and  $q$  be the other two prime numbers, we have:  $5pq = 5(p + q + 5)$

$$\Rightarrow pq - p - q + 1 = 6. \Rightarrow (p-1)(q-1) = 6 = 2 \times 3 = 1 \times 6.$$

If  $p-1 = 2$  and  $q-1 = 3$ ,  $q = 4$  is not a prime number which is not possible.

If  $p-1 = 1$  and  $q-1 = 6$ ,  $p = 2$  and  $q = 7$ . The answer is 7.

**Example 4.** (2015 AMC 10A) Consider the set of all fractions  $x/y$ ; where  $x$  and  $y$  are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 1, the value of the fraction is increased by 10%?

- (A) 0            (B) 1            (C) 2            (D) 3            (E) infinitely many

**Solution:** (B).

Method 1:

$$\begin{aligned} \frac{x}{y} + \frac{10x}{100y} &= \frac{x+1}{y+1} &\Rightarrow &\frac{x}{y} + \frac{x}{10y} = \frac{x+1}{y+1} &\Rightarrow &\frac{11x}{10y} = \frac{x+1}{y+1} \\ &\Rightarrow &11x(y+1) &= 10y(x+1) &\Rightarrow &xy + 11x - 10y = 0 \\ &\Rightarrow &(x-10)(y+11) &= -110 \end{aligned}$$

We know that  $y + 11$  is positive and  $y + 11 \geq 12$ .

So we have  $(x-10)(y+11) = -1 \times 110 = -2 \times 55 = -5 \times 22$ .

Case I:  $x-10 = -1$  and  $y+11 = 110 \Rightarrow (9, 99)$  ignored.

Case II:  $x-10 = -2$  and  $y+11 = 55 \Rightarrow (8, 44)$  ignored.

Case III:  $x-10 = -5$  and  $y+11 = 22 \Rightarrow (5, 11)$ .

So the answer is B.

Method 2:

$$\begin{aligned} \frac{x}{y} + \frac{10x}{100y} &= \frac{x+1}{y+1} &\Rightarrow &\frac{x}{y} + \frac{x}{10y} = \frac{x+1}{y+1} &\Rightarrow &\frac{11x}{10y} = \frac{x+1}{y+1} \\ &\Rightarrow &11x(y+1) &= 10y(x+1) &\Rightarrow &xy + 11x - 10y = 0 \\ \Rightarrow &y = \frac{11x}{10-x} &\Rightarrow &y = \frac{-11(10-x) + 110}{10-x} = \frac{11 \times 10}{10-x} - 11. \end{aligned}$$

We know that  $y > 0$ . So  $10 - x$  must be a factor of 10.

Case I:  $10 - x = 1 \Rightarrow x = 9$ , and  $y = 99$  (ignored).

Case II:  $10 - x = 2 \Rightarrow x = 8$ , and  $y = 44$  (ignored).

Case III:  $10 - x = 5 \Rightarrow x = 5$ , and  $y = 11$

So the answer is (B).

**Example 5.** (2002 AMC 10B) Find the value(s) of  $x$  such that  $8xy - 12y + 2x - 3 = 0$  is true for all values of  $y$ .

- (A)  $\frac{2}{3}$     (B)  $\frac{3}{2}$  or  $-\frac{1}{4}$     (C)  $-\frac{2}{3}$  or  $-\frac{1}{4}$     (D)  $\frac{3}{2}$     (E)  $-\frac{3}{2}$  or  $-\frac{1}{4}$

**Solution:** (D).

Method 1 (official solution):

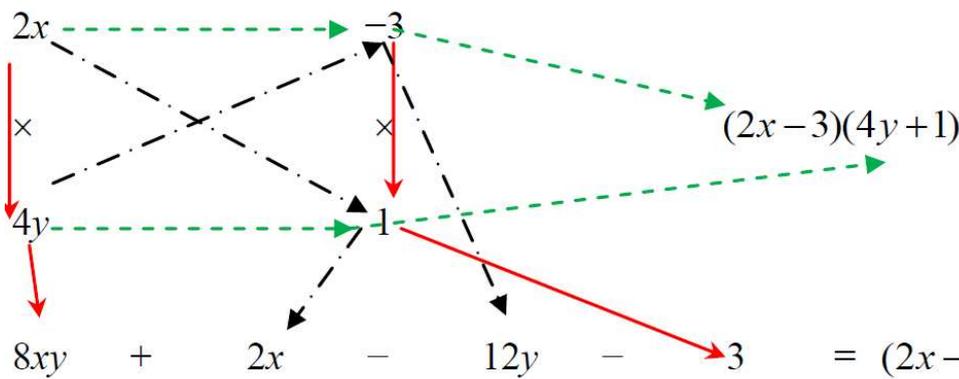
The given equation can be factored as

$$0 = 8xy - 12y + 2x - 3 = 4y(2x - 3) + (2x - 3) = (4y + 1)(2x - 3).$$

For this equation to be true for all values of  $y$  we must have  $2x - 3 = 0$ , that is,  $x = 3/2$ .

Method 2 (our solution):

$$8xy - 12y + 2x - 3 = 0$$



$$8xy + 2x - 12y - 3 = (2x-3)(4y+1)$$

$$8xy - 12y + 2x - 3 = (2x-3)(4y+1)$$

$$(2x-3)(4y+1) = 0.$$

For this equation to be true for all values of  $y$  we must have  $2x - 3 = 0$ , that is,  $x = 3/2$ .

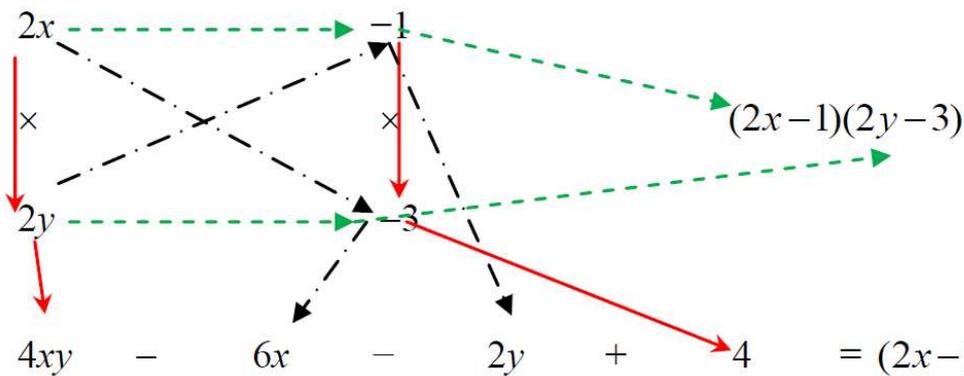
**Example 6.** How many ordered pairs  $(x, y)$  of integers are solutions to  $2xy - 3x - y = 6$ ?

- (A) 0      (B) 4      (C) 8      (D) 6      (E) 2

**Solution:** (C).

We multiply both sides of the given equation by 2:  $4xy - 6x - 2y = 12$

Then we factor  $4xy - 6x - 2y$ :



$$4xy - 6x - 2y + 4 = (2x-1)(2y-3)$$

$$4xy - 6x - 2y = 12 \Rightarrow (2x-1)(2y-3) - 4 = 11 \Rightarrow (2x-1)(2y-3) = 15.$$

So we have

- |   |   |
|---|---|
| 1. $\begin{cases} 2x-1=1 \\ 2y-3=15. \end{cases}$   | 2. $\begin{cases} 2x-1=15 \\ 2y-3=1. \end{cases}$   |
| 3. $\begin{cases} 2x-1=-1 \\ 2y-3=-15. \end{cases}$ | 4. $\begin{cases} 2x-1=-15 \\ 2y-3=-1. \end{cases}$ |

$$5. \begin{cases} 2x - 1 = 3 \\ 2y - 3 = 5. \end{cases}$$

$$6. \begin{cases} 2x - 1 = 5 \\ 2y - 3 = 3. \end{cases}$$

$$7. \begin{cases} 2x - 1 = -3 \\ 2y - 3 = -5. \end{cases}$$

$$8. \begin{cases} 2x - 1 = -5 \\ 2y - 3 = -3. \end{cases}$$

Solving we get:

$$\begin{cases} x = 1, \\ y = 9, \end{cases} \begin{cases} x = 8, \\ y = 2, \end{cases} \begin{cases} x = 0, \\ y = -6, \end{cases} \begin{cases} x = -7, \\ y = 1. \end{cases} \begin{cases} x = 2, \\ y = 4, \end{cases} \begin{cases} x = 3, \\ y = 3, \end{cases} \begin{cases} x = -1, \\ y = -1, \end{cases} \begin{cases} x = -2, \\ y = 0. \end{cases}$$

**Example 7.** (2000 AMC 10) Let  $A$ ,  $M$ , and  $C$  be nonnegative integers such that  $A + M + C = 10$ . What is the maximum value of  $A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A$ ?

- (A) 49      (B) 59      (C) 69      (D) 79      (E) 89

**Solution:** (C).

Method 1 (official solution):

Notice that  $A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A = (A + 1)(M + 1)(C + 1) - (A + M + C) - 1 = pqr - 11$ , where  $p$ ,  $q$ , and  $r$  are positive integers whose sum is 13. A simple case analysis shows that  $pqr$  is largest when two of the numbers  $p$ ,  $q$ ,  $r$  are 4 and the third is 5. Thus the answer is  $4 \cdot 4 \cdot 5 - 11 = 69$ .

Method 2 (our solutions):

We know that  $xyz + xy + yz + zx + x + y + z + 1 = (x + 1)(y + 1)(z + 1)$ .

So we have

$$A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A + (A + M + C) + 1 = (A + 1)(M + 1)(C + 1)$$

If  $(A + 1)(M + 1)(C + 1)$  has the maximum value, then  $A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A + (A + M + C) + 1$  has the maximum value or  $A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A$  has the maximum value.

$(A + 1)(M + 1)(C + 1)$  will have the maximum value if  $(A + 1)$ ,  $(M + 1)$ , and  $(C + 1)$  are as close as possible, or  $A$ ,  $M$ , and  $C$  are as close as possible.

Since  $A + M + C = 10$ ,  $A = 3$ ,  $M = 3$ , and  $C = 4$ .

Therefore  $A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A = 3 \times 3 \times 4 + 3 \times 3 + 3 \times 4 + 4 \times 3 = 36 + 9 + 12 + 12 = 69$ .

**Example 8.** Two non-zero real numbers,  $a$  and  $b$ , satisfy  $ab = a + b$ . Find a possible value  $ab - \left(\frac{a}{b} + \frac{b}{a}\right)$ .

- (A)  $-2$       (B)  $-\frac{1}{2}$       (C)  $\frac{1}{3}$       (D)  $\frac{1}{2}$       (E)  $2$

**Solution:** (E).

Find the common denominator and replace the  $ab$  in the numerator with  $a + b$  to

$$\text{get } ab - \left(\frac{a}{b} + \frac{b}{a}\right) = \frac{(ab)^2 - a^2 - b^2}{ab} = \frac{(a+b)^2 - a^2 - b^2}{ab}$$

$$= \frac{a^2 + 2ab + b^2 - a^2 - b^2}{ab} = \frac{2ab}{ab} = 2.$$

**Example 9.** What is the positive difference of two roots to the equation  $x^2 + |x| - 6 = 0$ ?

- (A)  $1$       (B)  $2$       (C)  $3$       (D)  $4$       (E)  $6$

**Solution:** (D).

We write the equation  $x^2 + |x| - 6 = 0$  as  $|x|^2 + |x| - 6 = 0$ .

We then factor the left hand side of the equation :  $(|x| + 3)(|x| - 2) = 0$

We know that  $(|x| + 3) > 0$ . So we have  $(|x| - 2) = 0 \Rightarrow |x| = 2 \Rightarrow x = \pm 2$ .

The positive difference of two roots is 4.

**Example 10.** Find the value of a positive integer  $x$  such that, when 64 is taken away from it, the result is a square number, and when 25 is added to it, the result is also a square number.

- (A) 2000      (B) 2016      (C) 3000      (D) 2025      (E) 1936

**Solution:** (A).

$$\left. \begin{array}{l} x - 64 = n^2 \\ x + 25 = m^2 \end{array} \right\} \Rightarrow m^2 - n^2 = 89 \Rightarrow (m - n)(m + n) = 89$$

Since 89 is a prime number and  $m + n > m - n$ ,

$$\left. \begin{array}{l} m + n = 89 \\ m - n = 1 \end{array} \right\} \Rightarrow m = 45, n = 44.$$

$$x = 45^2 - 25 = 2000.$$

**Example 11.** How many pairs  $(a, b)$ , where  $a$  and  $b$  are positive integers, satisfy the equation  $a^2 - b^2 = 2016$ ?

- (A) 4            (B) 7            (C) 10            (D) 12            (E) 30

**Solution:** (D).

$$(a-b)(a+b) = 2016 = 2^5 \times 3^2 \times 7.$$

We know  $a + b > a - b$  and  $a + b$  and  $a - b$  have the same parity. Since 2016 is an even number, both  $a + b$  and  $a - b$  must be even.

2016 has  $(4+1)(2+1)(1+1) = 30$  even factors and  $(2+1)(1+1) = 6$  odd factors. These 6 odd factors will pair with 6 even factors. So we have  $30 - 6 = 24$  even factors available. The number of pairs of  $a + b$  and  $a - b$  is then  $24 \div 2 = 12$  pairs. Thus the number of pairs of  $a$  and  $b$  is also 12.

**Example 12.** Find  $n$ , the number of all positive integer solutions to the equation  $x^2 - y^2 + 2y = 61$ .

- (A) 1            (B) 2            (C) 3            (D) 4            (E) 5

Completing the square yields

$$x^2 - (y-1)^2 = 60 \quad \Rightarrow \quad (x+y-1)(x-y+1) = 60.$$

We know that  $x + y - 1 \geq x - y + 1$ ,  $x + y - 1$  and  $x - y + 1$  have the same parity, and  $x - y + 1 > 0$ ,

$$\begin{cases} x + y - 1 = 30 \\ x - y + 1 = 2 \end{cases} \quad \Rightarrow \quad x = 16; \quad y = 15$$

$$\begin{cases} x + y - 1 = 10 \\ x - y + 1 = 6 \end{cases} \quad \Rightarrow \quad x = 8; \quad y = 3$$

Therefore, the two solutions for  $(x, y)$  are  $(16, 15)$  and  $(8, 3)$ .

**Example 13.** How many ordered triples  $(x, y, z)$  of nonnegative integers are solutions to  $4x^2 - 4y^2 - 4yz - z^2 = 96$ ?

- (A) 1            (B) 2            (C) 3            (D) 4            (E) more than 4

**Solution:** (E).

$$4x^2 - 4y^2 - 4yz - z^2 = 96 \Rightarrow 4x^2 - (4y^2 + 4yz + z^2) = 96 \Rightarrow$$

$$4x^2 - (2y + z)^2 = 96 \Rightarrow [2x + (2y + z)][2x - (2y + z)] = 2^5 \times 3.$$

We know that  $[2x + (2y + z)]$  is larger than  $[2x - (2y + z)]$  and they have the same parity. So

$$[2x + (2y + z)] = 48$$

$$[2x - (2y + z)] = 2$$

$x$  is not integer.

$$[2x + (2y + z)] = 24$$

$$[2x - (2y + z)] = 4$$

$x = 7$ .

$$[2x + (2y + z)] = 16$$

$$[2x - (2y + z)] = 6$$

$x$  is not integer.

$$[2x + (2y + z)] = 12$$

$$[2x - (2y + z)] = 8.$$

$x = 5$ .

$$\text{When } x = 5, 4x^2 - (2y + z)^2 = 96 \Rightarrow 100 - (2y + z)^2 = 96$$

$$\Rightarrow (2y + z)^2 = 4 \Rightarrow 2y + z = 2.$$

The solution are  $y = 1, z = 0$ ; and  $y = 0, z = 2$ .

$$\text{When } x = 7, 4x^2 - (2y + z)^2 = 96 \Rightarrow 196 - (2y + z)^2 = 96$$

$$\Rightarrow (2y + z)^2 = 100 \Rightarrow 2y + z = 10.$$

The solutions are  $y = 1, z = 8$ ;  $y = 2, z = 6$ ;  $y = 3, z = 4$ ;  $y = 4, z = 2$ ;  $y = 5, z = 0$ .

We get 7 triples  $(x, y, z)$  of nonnegative integers solutions.

**Example 14.** Find the value of  $b - c$  if  $a^3 = b^2$ ,  $c^2 = d$ , and  $d - a = 5$ , where  $a$ ,  $b$ ,  $c$ , and  $d$  are positive integers.

- (A) 1            (B) 2            (C) 3            (D) 4            (E) 5

**Solution:** (E).

Since  $a$  and  $b$  are positive integers, let  $a^3 = b^2 = t^{3 \times 2} = t^6$ , we get:  $a = t^2$  and  $b = t^3$ .

Therefore,  $d - a = 5$  can be written as  $c^2 - t^2 = 5$  or  $(c - t)(c + t) = 5$

Since 5 is a prime number and  $c + t > c - t$ , we have:

$$\left. \begin{array}{l} c + t = 5 \\ c - t = 1 \end{array} \right\} \Rightarrow c = 3, t = 2.$$

$$b - c = 2^3 - 3 = 5.$$

**Example 15.** (2002 AMC 10B) For how many positive integers  $n$  is  $n^2 - 3n + 2$  a prime number?

- (A) none      (B) one      (C) two      (D) more than two, but finitely many  
(E) infinitely many

**Solution:** (B).

Method 1 (official solution):

If  $n \geq 4$ , then  $n^2 - 3n + 2 = (n - 1)(n - 2)$  is the product of two integers greater than 1, and thus is not prime.

For  $n = 1, 2$ , and  $3$  we have, respectively,  $(1 - 1)(1 - 2) = 0$ ,  $(2 - 1)(2 - 2) = 0$ , and  $(3 - 1)(3 - 2) = 2$ .

Therefore,  $n^2 - 3n + 2$  is prime only when  $n = 3$ .

Method 2 (our solution):

We know that  $n^2 - 3n + 2 = (n - 1)(n - 2)$  is a prime if and only if

I:  $(n - 1) = 1$  and  $(n - 2)$  is a prime

$$n - 1 = 1 \Rightarrow n = 2 \text{ and } (n - 2) = 0 \text{ is not a prime.}$$

II:  $(n - 2) = 1$  and  $(n - 1)$  is a prime

$$n - 2 = 1 \Rightarrow n = 3 \text{ and } (n - 1) = 2 \text{ which is a prime.}$$

III:  $n - 1 = -1$  and  $(n - 2)$  is a negative value and  $|(n - 2)|$  is a prime.

$$n - 1 = -1 \Rightarrow n = 0 \text{ (this case is ignored since } n \text{ is a positive integer).}$$

IV:  $(n - 2) = -1$  and  $(n - 1)$  is a negative value and  $|(n - 1)|$  is a prime.

$$(n - 2) = -1 \Rightarrow n = 1 \text{ and } (n - 1) = 0 \text{ (this case is ignored since } n \text{ is a positive integer).}$$

Therefore,  $n^2 - 3n + 2$  is prime only when  $n = 3$ .

**Example 16.** For how many integers  $n$  is  $n^2 - 4n - 21$  a prime number?

- (A) 0            (B) 1            (C) 2            (D) 5            (E) 7

**Solution:** (C).

We know that  $n^2 - 4n - 21 = (n + 3)(n - 7)$  is a prime if and only if

I:  $(n + 3) = 1$  and  $(n - 7)$  is a prime

$$n + 3 = 1 \quad \Rightarrow \quad n = -2.$$

$(n - 7) = -9$  is a prime or not.

II:  $(n - 7) = 1$  and  $(n + 3)$  is a prime

$$n - 7 = 1 \quad \Rightarrow \quad n = 8 \text{ and } (n + 3) = 11 \text{ which is a prime.}$$

III:  $(n + 3) = -1$  and  $(n - 7)$  is a negative value and  $|(n - 7)|$  is a prime.

$$n + 3 = -1 \quad \Rightarrow \quad n = -4 \text{ and } |(n - 7)| = 11 \text{ is a prime.}$$

IV:  $(n - 7) = -1$  and  $(n + 3)$  is a negative value and  $|(n + 3)|$  is a prime.

$$n - 7 = -1 \quad \Rightarrow \quad n = 6 \text{ and } (n + 3) = 9 \text{ which is not a prime.}$$

Therefore,  $n^2 - 4n - 21$  is prime only when  $n = 8$  and  $n = -4$ .

**Example 17.** (2005 AMC 10B Problem 24) Let  $x$  and  $y$  be two-digit integers such that  $y$  is obtained by reversing the digits of  $x$ . The integers  $x$  and  $y$  satisfy  $x^2 - y^2 = m^2$  for some positive integer  $m$ . What is  $x + y + m$ ?

- (A) 88            (B) 112            (C) 116            (D) 144            (E) 154

**Solution:** (E).

Method 1 (official solution):

By the given conditions, it follows that  $x > y$ .

Let  $x = 10a + b$  and  $y = 10b + a$ , where  $a > b$ . Then

$$m^2 = x^2 - y^2 = (10a + b)^2 - (10b + a)^2 = 99a^2 - 99b^2 = 99(a^2 - b^2).$$

Since  $99(a^2 - b^2)$  must be a perfect square,

$$a^2 - b^2 = (a + b)(a - b) = 11k^2 \text{ for some positive integer } k.$$

Because  $a$  and  $b$  are distinct digits, we have  $a - b \leq 9 - 1 = 8$  and  $a + b \leq 9 + 8 =$

17. It follows that  $a + b = 11$ ,  $a - b = k^2$ , and  $k$  is either 1 or 2.

If  $k = 2$ , then  $(a, b) = (15/2, 7/2)$ , which is impossible. Thus  $k = 1$  and  $(a, b) = (6, 5)$ . This gives  $x = 65$ ,  $y = 56$ ,  $m = 33$ , and  $x + y + m = 154$ .

Method 2 (our solution):

By the given conditions, it follows that  $x > y$ .

Let  $x = 10a + b$  and  $y = 10b + a$ , where  $a > b$ . Then

$$m^2 = x^2 - y^2 = (10a + b)^2 - (10b + a)^2 = 99a^2 - 99b^2 = 99(a^2 - b^2).$$

Since  $99(a^2 - b^2)$  must be a perfect square, for some positive integer  $n$ , we have

$$a^2 - b^2 = (a + b)(a - b) = 11n^2 \tag{1}$$

Note that  $(a + b)$  and  $(a - b)$  have the same parity.

We have the following two cases:

Case I:

$$\left. \begin{aligned} a + b &= 11 \\ a - b &= n^2 \end{aligned} \right\} \tag{2}$$

or

$$\left. \begin{aligned} a + b &= n^2 \\ a - b &= 11 \end{aligned} \right\} \tag{3}$$

Both systems give us  $a = \frac{11 + n^2}{2}$ .

$n$  is odd and  $a$  is the digit from 6 to 9. Thus  $6 \leq \frac{11 + n^2}{2} \leq 9 \Rightarrow 1 \leq n^2 \leq 7$ .

The only value for  $n$  is 1, and so  $a = 6$ . Plugging this value into (2), we get  $b = 5$ . This results in  $x = 65$ ,  $y = 56$ ,  $m = 33$ , and  $x + y + m = 154$ .

Case II:

$$\left. \begin{aligned} a + b &= n \\ a - b &= 11n \end{aligned} \right\} \tag{4}$$

or

$$\left. \begin{aligned} a + b &= 11n \\ a - b &= n \end{aligned} \right\} \tag{5}$$

Both systems give us  $a = 6n$ .

We know that  $a$  is a nonzero digit less than 9, and so  $n = 1$  and  $a = 6$ .

Substituting this into (5), we have  $b = 5$ .

This gives us  $x = 65$ ,  $y = 56$ ,  $m = 33$ , and  $x + y + m = 154$ .

