Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2007 Euclid Contest

Tuesday, April 17, 2007

Solutions

1. (a) Since $(a-1, a+1)$ lies on the line $y = 2x - 3$, then $a + 1 = 2(a-1) - 3$ or $a + 1 = 2a - 5$ or $a = 6$.

   (b) *Solution 1*

   To get from P to Q, we move 3 units right and 4 units up.

   Since $PQ = QR$ and R lies on the line through Q, then we must use the same motion to get from Q to R.

   Therefore, to get from $Q(0,4)$ to R, we move 3 units right and 4 units up, so the coordinates of R are $(3,8)$.

   Solution 2

   The line through $P(-3,0)$ and $Q(0,4)$ has slope $\dfrac{4-0}{0-(-3)} = \dfrac{4}{3}$ and y-intercept 4, so has equation $y = \frac{4}{3}x + 4$.

   Thus, R has coordinates $(a, \frac{4}{3}a + 4)$ for some $a > 0$.

   Since $PQ = QR$, then $PQ^2 = QR^2$, so

$$\begin{aligned} (-3)^2 + 4^2 &= a^2 + \left(\tfrac{4}{3}a + 4 - 4\right)^2 \\ 25 &= a^2 + \tfrac{16}{9}a^2 \\ \tfrac{25}{9}a^2 &= 25 \\ a^2 &= 9 \end{aligned}$$

   so $a = 3$ since $a > 0$.

   Thus, R has coordinates $(3, \frac{4}{3}(3) + 4) = (3,8)$.

   (c) Since $OP = 9$, then the coordinates of P are $(9,0)$.

   Since $OP = 9$ and $OA = 15$, then by the Pythagorean Theorem,

$$AP^2 = OA^2 - OP^2 = 15^2 - 9^2 = 144$$

   so $AP = 12$.

   Since P has coordinates $(9,0)$ and A is 12 units directly above P, then A has coordinates $(9,12)$.

   Since $PB = 4$, then B has coordinates $(13,0)$.

   The line through $A(9,12)$ and $B(13,0)$ has slope $\dfrac{12-0}{9-13} = -3$ so, using the point-slope form, has equation $y - 0 = -3(x - 13)$ or $y = -3x + 39$.

2. (a) Since $\cos(\angle BAC) = \dfrac{AB}{AC}$ and $\cos(\angle BAC) = \frac{5}{13}$ and $AB = 10$, then $AC = \frac{13}{5}AB = 26$.

   Since $\triangle ABC$ is right-angled at B, then by the Pythagorean Theorem,

   $BC^2 = AC^2 - AB^2 = 26^2 - 10^2 = 576$ so $BC = 24$ since $BC > 0$.

   Therefore, $\tan(\angle ACB) = \dfrac{AB}{BC} = \dfrac{10}{24} = \dfrac{5}{12}$.

(b) Since $2\sin^2 x + \cos^2 x = \frac{25}{16}$ and $\sin^2 x + \cos^2 x = 1$ (so $\cos^2 x = 1 - \sin^2 x$), then we get

$$\begin{aligned} 2\sin^2 x + (1 - \sin^2 x) &= \tfrac{25}{16} \\ \sin^2 x &= \tfrac{25}{16} - 1 \\ \sin^2 x &= \tfrac{9}{16} \\ \sin x &= \pm\tfrac{3}{4} \end{aligned}$$

so $\sin x = \frac{3}{4}$ since $\sin x > 0$ because $0^\circ < x < 90^\circ$.

(c) Since $\triangle ABC$ is isosceles and right-angled, then $\angle BAC = 45^\circ$.
Also, $AC = \sqrt{2}AB = \sqrt{2}(2\sqrt{2}) = 4$.
Since $\angle EAB = 75^\circ$ and $\angle BAC = 45^\circ$, then $\angle CAE = \angle EAB - \angle BAC = 30^\circ$.
Since $\triangle AEC$ is right-angled and has a 30° angle, then $\triangle AEC$ is a 30°-60°-90° triangle.
Thus, $EC = \frac{1}{2}AC = 2$ (since EC is opposite the 30° angle) and $AE = \frac{\sqrt{3}}{2}AC = 2\sqrt{3}$ (since AE is opposite the 60° angle).
In $\triangle CDE$, $ED = DC$ and $\angle EDC = 60^\circ$, so $\triangle CDE$ is equilateral.
Therefore, $ED = CD = EC = 2$.
Overall, the perimeter of $ABCDE$ is

$$AB + BC + CD + DE + EA = 2\sqrt{2} + 2\sqrt{2} + 2 + 2 + 2\sqrt{3} = 4 + 4\sqrt{2} + 2\sqrt{3}$$

3. (a) From the given information, the first term in the sequence is 2007 and each term starting with the second can be determined from the previous term.
The second term is $2^3 + 0^3 + 0^3 + 7^3 = 8 + 0 + 0 + 343 = 351$.
The third term is $3^3 + 5^3 + 1^3 = 27 + 125 + 1 = 153$.
The fourth term is $1^3 + 5^3 + 3^3 = 27 + 125 + 1 = 153$.
Since two consecutive terms are equal, then every term thereafter will be equal, because each term depends only on the previous term and a term of 153 always makes the next term 153.
Thus, the 2007th term will be 153.

(b) The nth term of sequence A is $n^2 - 10n + 70$.
Since sequence B is arithmetic with first term 5 and common difference 10, then the nth term of sequence B is equal to $5 + 10(n-1) = 10n - 5$. (Note that this formula agrees with the first few terms.)
For the nth term of sequence A to be equal to the nth term of sequence B, we must have

$$\begin{aligned} n^2 - 10n + 70 &= 10n - 5 \\ n^2 - 20n + 75 &= 0 \\ (n-5)(n-15) &= 0 \end{aligned}$$

Therefore, $n = 5$ or $n = 15$. That is, 5th and 15th terms of sequence A and sequence B are equal to each other.

4. (a) *Solution 1*

Rearranging and then squaring both sides,

$$\begin{aligned} 2+\sqrt{x-2} &= x-2 \\ \sqrt{x-2} &= x-4 \\ x-2 &= (x-4)^2 \\ x-2 &= x^2-8x+16 \\ 0 &= x^2-9x+18 \\ 0 &= (x-3)(x-6) \end{aligned}$$

so $x=3$ or $x=6$.

We should check both solutions, because we may have introduced extraneous solutions by squaring.

If $x=3$, the left side equals $2+\sqrt{1}=3$ and the right side equals 1, so $x=3$ must be rejected.

If $x=6$, the left side equals $2+\sqrt{4}=4$ and the right side equals 4, so $x=6$ is the only solution.

Solution 2

Suppose $u=\sqrt{x-2}$.

The equation becomes $2+u=u^2$ or $u^2-u-2=0$ or $(u-2)(u+1)=0$.

Therefore, $u=2$ or $u=-1$.

But we cannot have $\sqrt{x-2}=-1$ (as square roots are always non-negative).

Therefore, $\sqrt{x-2}=2$ or $x-2=4$ or $x=6$.

We can check as in Solution 1 that $x=6$ is indeed a solution.

(b) *Solution 1*

From the diagram, the parabola has x-intercepts $x=3$ and $x=-3$.

Therefore, the equation of the parabola is of the form $y=a(x-3)(x+3)$ for some real number a.

Triangle ABC can be considered as having base AB (of length $3-(-3)=6$) and height OC (where O is the origin).

Suppose C has coordinates $(0,-c)$. Then $OC=c$.

Thus, the area of $\triangle ABC$ is $\frac{1}{2}(AB)(OC)=3c$. But we know that the area of $\triangle ABC$ is 54, so $3c=54$ or $c=18$.

Since the parabola passes through $C(0,-18)$, then this point must satisfy the equation of the parabola.

Therefore, $-18=a(0-3)(0+3)$ or $-18=-9a$ or $a=2$.

Thus, the equation of the parabola is $y=2(x-3)(x+3)=2x^2-18$.

Solution 2

Triangle ABC can be considered as having base AB (of length $3-(-3)=6$) and height OC (where O is the origin).

Suppose C has coordinates $(0,-c)$. Then $OC=c$.

Thus, the area of $\triangle ABC$ is $\frac{1}{2}(AB)(OC)=3c$. But we know that the area of $\triangle ABC$ is 54, so $3c=54$ or $c=18$.

Therefore, the parabola has vertex $C(0,-18)$, so has equation $y=a(x-0)^2-18$.

(The vertex of the parabola must lie on the y-axis since its roots are equally distant from the y-axis, so C must be the vertex.)

Since the parabola passes through $B(3,0)$, then these coordinates satisfy the equation, so $0=3^2a-18$ or $9a=18$ or $a=2$.

Therefore, the equation of the parabola is $y=2x^2-18$.

5. (a) The perimeter of the sector is made up of two line segments (of total length $5+5=10$) and one arc of a circle.

Since $\dfrac{72^\circ}{360^\circ}=\dfrac{1}{5}$, then the length of the arc is $\dfrac{1}{5}$ of the total circumference of a circle of radius 5.

Thus, the length of the arc is $\frac{1}{5}(2\pi(5))=2\pi$.

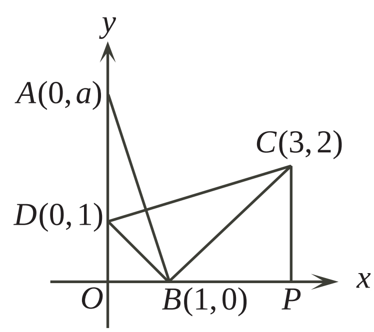
Therefore, the perimeter of the sector is $10+2\pi$.

(b) $\triangle AOB$ is right-angled at O, so has area $\frac{1}{2}(AO)(OB)=\frac{1}{2}a(1)=\frac{1}{2}a$.

We next need to calculate the area of $\triangle BCD$.

Method 1: Completing the trapezoid

Drop a perpendicular from C to $P(3,0)$ on the x-axis.



Then $DOPC$ is a trapezoid with parallel sides DO of length 1 and PC of length 2 and height OP (which is indeed perpendicular to the parallel sides) of length 3.

The area of the trapezoid is thus $\frac{1}{2}(DO+PC)(OP)=\frac{1}{2}(1+2)(3)=\frac{9}{2}$.

But the area of $\triangle BCD$ equals the area of trapezoid $DOPC$ minus the areas of $\triangle DOB$ and $\triangle BPC$.

$\triangle DOB$ is right-angled at O, so has area $\frac{1}{2}(DO)(OB)=\frac{1}{2}(1)(1)=\frac{1}{2}$.

$\triangle BPC$ is right-angled at P, so has area $\frac{1}{2}(BP)(PC)=\frac{1}{2}(2)(2)=2$.

Thus, the area of $\triangle DBC$ is $\frac{9}{2}-\frac{1}{2}-2=2$.

(A similar method for calculating the area of $\triangle DBC$ would be to drop a perpendicular to Q on the y-axis, creating a rectangle $QOPC$.)

Method 2: $\triangle DBC$ is right-angled

The slope of line segment DB is $\dfrac{1-0}{0-1} = -1$.

The slope of line segment BC is $\dfrac{2-0}{3-1} = 1$.

Since the product of these slopes is -1 (that is, their slopes are negative reciprocals), then DB and BC are perpendicular.

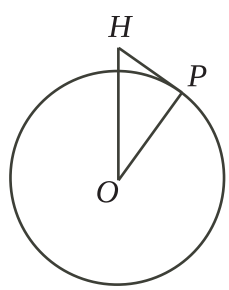
Therefore, the area of $\triangle DBC$ is $\frac{1}{2}(DB)(BC)$.

Now $DB = \sqrt{(1-0)^2 + (0-1)^2} = \sqrt{2}$ and $BC = \sqrt{(3-1)^2 + (2-0)^2} = \sqrt{8}$.

Thus, the area of $\triangle DBC$ is $\frac{1}{2}\sqrt{2}\sqrt{8} = 2$.

Since the area of $\triangle AOB$ equals the area of $\triangle DBC$, then $\frac{1}{2}a = 2$ or $a = 4$.

6. (a) Suppose that O is the centre of the planet, H is the place where His Highness hovers in the helicopter, and P is the furthest point on the surface of the planet that he can see.



Then HP must be a tangent to the surface of the planet (otherwise he could see further), so OP (a radius) is perpendicular to HP (a tangent).

We are told that $OP = 24$ km.

Since the helicopter hovers at a height of 2 km, then $OH = 24 + 2 = 26$ km.

Therefore, $HP^2 = OH^2 - OP^2 = 26^2 - 24^2 = 100$, so $HP = 10$ km.

Therefore, the distance to the furthest point that he can see is 10 km.

(b) Since we know the measure of $\angle ADB$, then to find the distance AB, it is enough to find the distances AD and BD and then apply the cosine law.

In $\triangle DBE$, we have $\angle DBE = 180^\circ - 20^\circ - 70^\circ = 90^\circ$, so $\triangle DBE$ is right-angled, giving $BD = 100\cos(20^\circ) \approx 93.969$.

In $\triangle DAC$, we have $\angle DAC = 180^\circ - 50^\circ - 45^\circ = 85^\circ$.

Using the sine law, $\dfrac{AD}{\sin(50^\circ)} = \dfrac{CD}{\sin(85^\circ)}$, so $AD = \dfrac{150\sin(50^\circ)}{\sin(85^\circ)} \approx 115.346$.

Finally, using the cosine law in $\triangle ABD$, we get

$$\begin{aligned} AB^2 &= AD^2 + BD^2 - 2(AD)(BD)\cos(\angle ADB) \\ AB^2 &\approx (115.346)^2 + (93.969)^2 - 2(115.346)(93.969)\cos(35^\circ) \\ AB^2 &\approx 4377.379 \\ AB &\approx 66.16 \end{aligned}$$

Therefore, the distance from A to B is approximately 66 m.

7. (a) Using rules for manipulating logarithms,

$$\begin{aligned} (\sqrt{x})^{\log_{10} x} &= 100 \\ \log_{10}\left((\sqrt{x})^{\log_{10} x}\right) &= \log_{10} 100 \\ (\log_{10} x)(\log_{10}\sqrt{x}) &= 2 \\ (\log_{10} x)(\log_{10} x^{\frac{1}{2}}) &= 2 \\ (\log_{10} x)(\tfrac{1}{2}\log_{10} x) &= 2 \\ (\log_{10} x)^2 &= 4 \\ \log_{10} x &= \pm 2 \\ x &= 10^{\pm 2} \end{aligned}$$

Therefore, $x = 100$ or $x = \frac{1}{100}$.
(We can check by substitution that each is indeed a solution.)

(b) *Solution 1*

Without loss of generality, suppose that square $ABCD$ has side length 1.
Suppose next that $BF = a$ and $\angle CFB = \theta$.
Since $\triangle CBF$ is right-angled at B, then $\angle BCF = 90^\circ - \theta$.
Since GCF is a straight line, then $\angle GCD = 180^\circ - 90^\circ - (90^\circ - \theta) = \theta$.
Therefore, $\triangle GDC$ is similar to $\triangle CBF$, since $\triangle GDC$ is right-angled at D.
Thus, $\dfrac{GD}{DC} = \dfrac{BC}{BF}$ or $\dfrac{GD}{1} = \dfrac{1}{a}$ or $GD = \dfrac{1}{a}$.
So $AF = AB + BF = 1 + a$ and $AG = AD + DG = 1 + \dfrac{1}{a} = \dfrac{a+1}{a}$.
Thus, $\dfrac{1}{AF} + \dfrac{1}{AG} = \dfrac{1}{1+a} + \dfrac{a}{a+1} = \dfrac{a+1}{a+1} = 1 = \dfrac{1}{AB}$, as required.

Solution 2

We attach a set of coordinate axes to the diagram, with A at the origin, AG lying along the positive y-axis and AF lying along the positive x-axis.
Without loss of generality, suppose that square $ABCD$ has side length 1, so that C has coordinates $(1, 1)$. (We can make this assumption without loss of generality, because if the square had a different side length, then each of the lengths in the problem would be scaled by the same factor.)

Suppose that the line through G and F has slope m.
Since this line passes through $(1,1)$, its equation is $y-1=m(x-1)$ or $y=mx+(1-m)$.
The y-intercept of this line is $1-m$, so G has coordinates $(0,1-m)$.
The x-intercept of this line is $\frac{m-1}{m}$, so F has coordinates $\left(\frac{m-1}{m},0\right)$. (Note that $m\neq 0$ as the line cannot be horizontal.)
Therefore,

$$\frac{1}{AF}+\frac{1}{AG}=\frac{m}{m-1}+\frac{1}{1-m}=\frac{m}{m-1}+\frac{-1}{m-1}=\frac{m-1}{m-1}=1=\frac{1}{AB}$$

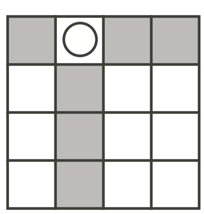
as required.

Solution 3
Join A to C.
We know that the sum of the areas of $\triangle GCA$ and $\triangle FCA$ equals the area of $\triangle GAF$.
The area of $\triangle GCA$ (thinking of AG as the base) is $\frac{1}{2}(AG)(DC)$, since DC is perpendicular to AG.
Similarly, the area of $\triangle FCA$ is $\frac{1}{2}(AF)(CB)$.
Also, the area of $\triangle GAF$ is $\frac{1}{2}(AG)(AF)$.
Therefore,

$$\begin{aligned}\tfrac{1}{2}(AG)(DC)+\tfrac{1}{2}(AF)(CB) &= \tfrac{1}{2}(AG)(AF)\\ \frac{(AG)(DC)}{(AG)(AF)(AB)}+\frac{(AF)(CB)}{(AG)(AF)(AB)} &= \frac{(AG)(AF)}{(AG)(AF)(AB)}\\ \frac{1}{AF}+\frac{1}{AG} &= \frac{1}{AB}\end{aligned}$$

as required, since $AB=DC=CB$.

8. (a) We consider placing the three coins individually.
Place one coin randomly on the grid.
When the second coin is placed (in any one of 15 squares), 6 of the 15 squares will leave two coins in the same row or column and 9 of the 15 squares will leave the two coins in different rows and different columns.



Therefore, the probability that the two coins are in different rows and different columns is $\frac{9}{15}=\frac{3}{5}$.
There are 14 possible squares in which the third coin can be placed.

Of these 14 squares, 6 lie in the same row or column as the first coin and an additional 4 lie the same row or column as the second coin. Therefore, the probability that the third coin is placed in a different row and a different column than each of the first two coins is $\frac{4}{14} = \frac{2}{7}$.

Therefore, the probability that all three coins are placed in different rows and different columns is $\frac{3}{5} \times \frac{2}{7} = \frac{6}{35}$.

(b) Suppose that $AB = c$, $AC = b$ and $BC = a$.

Since DG is parallel to AC, $\angle BDG = \angle BAC$ and $\angle DGB = \angle ACB$, so $\triangle DGB$ is similar to $\triangle ACB$.

(Similarly, $\triangle AED$ and $\triangle ECF$ are also both similar to $\triangle ABC$.)

Suppose next that $DB = kc$, with $0 < k < 1$.

Then the ratio of the side lengths of $\triangle DGB$ to those of $\triangle ACB$ will be $k : 1$, so $BG = ka$ and $DG = kb$.

Since the ratio of the side lengths of $\triangle DGB$ to $\triangle ACB$ is $k : 1$, then the ratio of their areas will be $k^2 : 1$, so the area of $\triangle DGB$ is k^2 (since the area of $\triangle ACB$ is 1).

Since $AB = c$ and $DB = kc$, then $AD = (1-k)c$, so using similar triangles as before, $DE = (1-k)a$ and $AE = (1-k)b$. Also, the area of $\triangle ADE$ is $(1-k)^2$.

Since $AC = b$ and $AE = (1-k)b$, then $EC = kb$, so again using similar triangles, $EF = kc$, $FC = ka$ and the area of $\triangle ECF$ is k^2.

Now the area of trapezoid $DEFG$ is the area of the large triangle minus the combined areas of the small triangles, or $1 - k^2 - k^2 - (1-k)^2 = 2k - 3k^2$.

We know that $k \geq 0$ by its definition. Also, since G is to the left of F, then $BG + FC \leq BC$ or $ka + ka \leq a$ or $2ka \leq a$ or $k \leq \frac{1}{2}$.

Let $f(k) = 2k - 3k^2$.

Since $f(k) = -3k^2 + 2k + 0$ is a parabola opening downwards, its maximum occurs at its vertex, whose k-coordinate is $k = -\frac{2}{2(-3)} = \frac{1}{3}$ (which lies in the admissible range for k).

Note that $f(\frac{1}{3}) = \frac{2}{3} - 3(\frac{1}{9}) = \frac{1}{3}$.

Therefore, the maximum area of the trapezoid is $\frac{1}{3}$.

9. (a) The vertex of the first parabola has x-coordinate $x = -\frac{1}{2}b$.

Since each parabola passes through P, then

$$\begin{aligned}
f\left(-\tfrac{1}{2}b\right) &= g\left(-\tfrac{1}{2}b\right) \\
\tfrac{1}{4}b^2 + b(-\tfrac{1}{2}b) + c &= -\tfrac{1}{4}b^2 + d(-\tfrac{1}{2}b) + e \\
\tfrac{1}{4}b^2 - \tfrac{1}{2}b^2 + c &= -\tfrac{1}{4}b^2 - \tfrac{1}{2}bd + e \\
\tfrac{1}{2}bd &= e - c \\
bd &= 2(e-c)
\end{aligned}$$

as required. (The same result can be obtained by using the vertex of the second parabola.)

(b) *Solution 1*

The vertex, P, of the first parabola has x-coordinate $x = -\frac{1}{2}b$ so has y-coordinate $f(-\frac{1}{2}b) = \frac{1}{4}b^2 - \frac{1}{2}b^2 + c = -\frac{1}{4}b^2 + c$.

The vertex, Q, of the first parabola has x-coordinate $x = \frac{1}{2}d$ so has y-coordinate $g(\frac{1}{2}d) = -\frac{1}{4}d^2 + \frac{1}{2}d^2 + c = \frac{1}{4}d^2 + e$.

Therefore, the slope of the line through P and Q is

$$\begin{aligned}
\frac{(-\frac{1}{4}b^2 + c) - (\frac{1}{4}d^2 + e)}{-\frac{1}{2}b - \frac{1}{2}d} &= \frac{-\frac{1}{4}(b^2 + d^2) - (e - c)}{-\frac{1}{2}b - \frac{1}{2}d} \\
&= \frac{-\frac{1}{4}(b^2 + d^2) - \frac{1}{2}bd}{-\frac{1}{2}b - \frac{1}{2}d} \\
&= \frac{-\frac{1}{4}(b^2 + 2bd + d^2)}{-\frac{1}{2}(b + d)} \\
&= \frac{1}{2}(b + d)
\end{aligned}$$

Using the point-slope form of the line, the line thus has equation

$$\begin{aligned}
y &= \tfrac{1}{2}(b + d)(x - (-\tfrac{1}{2}b)) + (-\tfrac{1}{4}b^2 + c) \\
&= \tfrac{1}{2}(b + d)x + \tfrac{1}{4}b^2 + \tfrac{1}{4}bd - \tfrac{1}{4}b^2 + c \\
&= \tfrac{1}{2}(b + d)x + \tfrac{1}{4}bd + c \\
&= \tfrac{1}{2}(b + d)x + \tfrac{1}{2}(e - c) + c \\
&= \tfrac{1}{2}(b + d)x + \tfrac{1}{2}(e + c)
\end{aligned}$$

so the y-intercept of the line is $\frac{1}{2}(e + c)$.

Solution 2

The equations of the two parabolas are $y = x^2 + bx + c$ and $y = -x^2 + dx + e$.

Adding the two equations, we obtain $2y = (b + d)x + (c + e)$ or $y = \frac{1}{2}(b + d)x + \frac{1}{2}(c + e)$.

This last equation is the equation of a line.

Points P and Q, whose coordinates satisfy the equation of each parabola, must satisfy the equation of the line, and so lie on the line.

But the line through P and Q is unique, so this is the equation of the line through P and Q.

Therefore, the line through P and Q has slope $\frac{1}{2}(b + d)$ and y-intercept $\frac{1}{2}(c + e)$.

10. (a) First, we note that since the circle and lines XY and XZ are fixed, then the quantity $XY + XZ$ is fixed.

Since VT and VY are tangents from the same point V to the circle, then $VT = VY$.

Since WT and WZ are tangents from the same point W to the circle, then $WT = WZ$.

Therefore, the perimeter of $\triangle VXW$ is

$$\begin{aligned} XV+XW+VW &= XV+XW+VT+WT \\ &= XV+XW+VY+WZ \\ &= XV+VY+XW+WZ \\ &= XY+XZ \end{aligned}$$

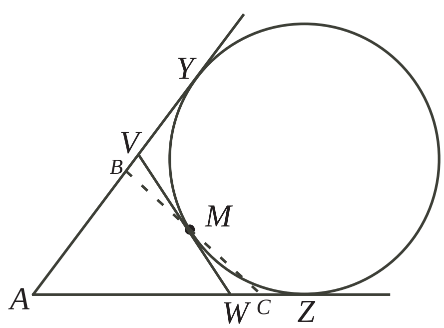
which is constant, by our earlier comment.

Therefore, the perimeter of $\triangle VXW$ always equals $XY+XZ$, which does not depend on the position of T.

(b) *Solution 1*

A circle can be drawn that is tangent to the lines AB extended and AC extended, that passes through M, and that has M on the left side of the circle. (The fact that such a circle can be drawn and that this circle is unique can be seen by starting with a small circle tangent to the two lines and expanding the circle, keeping it tangent to the two lines, until it has M on the left side of its circumference.) Suppose that this circle is tangent to AB and AC extended at Y and Z, respectively.

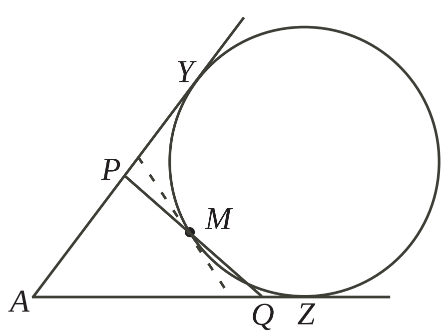
Draw a line tangent to the circle at M that cuts AB (extended) at V and AC (extended) at W.



We prove that $\triangle AVW$ has the minimum perimeter of all triangles that can be drawn with their third side passing through M.

From (a), we know that the perimeter of $\triangle AVW$ equals $AY+AZ$.

Consider a different triangle APQ formed by drawing another line through M. Note that this line PMQ cannot be tangent to the circle, so must cut the circle in two places (at M and at another point).



This line, however, will be tangent to a new circle that is tangent to AB and AC at Y' and Z'. But PMQ cuts the original circle at two points, then this new circle must be formed by shifting the original circle to the right. In other words, Y' and Z' will be further along AB and AC than Y and Z.
But the perimeter of $\triangle APQ$ will equal $AY' + AZ'$ by (a) and $AY' + AZ' > AY + AZ$, so the perimeter of $\triangle APQ$ is greater than that of $\triangle AVW$.
Therefore, the perimeter is minimized when the line through M is tangent to the circle.

We now must determine the perimeter of $\triangle AVW$. Note that it is sufficient to determine the length of AZ, since the perimeter of $\triangle AVW$ equals $AY + AZ$ and $AY = AZ$, so the perimeter of $\triangle AVW$ is twice the length of AZ.
First, we calculate $\angle VAW = \angle BAC$ using the cosine law:

$$\begin{aligned}
BC^2 &= AB^2 + AC^2 - 2(AB)(AC)\cos(\angle BAC) \\
14^2 &= 10^2 + 16^2 - 2(10)(16)\cos(\angle BAC) \\
196 &= 356 - 320\cos(\angle BAC) \\
320\cos(\angle BAC) &= 160 \\
\cos(\angle BAC) &= \tfrac{1}{2} \\
\angle BAC &= 60^\circ
\end{aligned}$$

Next, we add coordinates to the diagram by placing A at the origin $(0,0)$ and AC along the positive x-axis. Thus, C has coordinates $(16,0)$.
Since $\angle BAC = 60^\circ$ and $AB = 10$, then B has coordinates $(10\cos(60^\circ), 10\sin(60^\circ))$ or $(5, 5\sqrt{3})$.
Since M is the midpoint of BC, then M has coordinates $\left(\frac{1}{2}(5+16), \frac{1}{2}(5\sqrt{3}+0)\right)$ or $\left(\frac{21}{2}, \frac{5}{2}\sqrt{3}\right)$.
Suppose the centre of the circle is O and the circle has radius r.
Since the circle is tangent to the two lines AY and AZ, then the centre of the circle lies on the angle bisector of $\angle BAC$, so lies on the line through the origin that makes an angle of 30° with the positive x-axis. The slope of this line is thus $\tan(30^\circ) = \frac{1}{\sqrt{3}}$.
The centre O will have y-coordinate r, since a radius from the centre to AZ is perpendicular to the x-axis. Thus, O has coordinates $(\sqrt{3}r, r)$ and Z has coordinates $(\sqrt{3}r, 0)$.
Thus, the perimeter of the desired triangle is $2AZ = 2\sqrt{3}r$.
Since the circle has centre $(\sqrt{3}r, r)$ and radius r, then its equation is
$(x - \sqrt{3}r)^2 + (y - r)^2 = r^2$.
Since M lies on the circle, then when we substitute the coordinates of M, we obtain an

equation for r:

$$\begin{aligned}
\left(\tfrac{21}{2}-\sqrt{3}r\right)^2+\left(\tfrac{5}{2}\sqrt{3}-r\right)^2 &= r^2 \\
\tfrac{441}{4}-21\sqrt{3}r+3r^2+\tfrac{75}{4}-5\sqrt{3}r+r^2 &= r^2 \\
3r^2-26\sqrt{3}r+129 &= 0 \\
(\sqrt{3}r)^2-2(13)(\sqrt{3}r)+169-40 &= 0 \\
(\sqrt{3}r-13)^2 &= 40 \\
\sqrt{3}r-13 &= \pm 2\sqrt{10} \\
r &= \frac{13\pm 2\sqrt{10}}{\sqrt{3}} \\
r &= \frac{13\sqrt{3}\pm 2\sqrt{30}}{3}
\end{aligned}$$

(Alternatively, we could have used the quadratic formula instead of completing the square.)

Therefore, $r=\dfrac{13\sqrt{3}+2\sqrt{30}}{3}$ since we want the circle with the larger radius that passes through M and is tangent to the two lines. (Note that there is a smaller circle "inside" M and a larger circle "outside" M.)

Therefore, the minimum perimeter is $2\sqrt{3}r=\dfrac{26(3)+4\sqrt{90}}{3}=26+4\sqrt{10}$.

Solution 2

As in Solution 1, we prove that the triangle with minimum perimeter has perimeter equal to $AY+AZ$.

Next, we must determine the length of AY.

As in Solution 1, we can show that $\angle YAZ=60^\circ$.

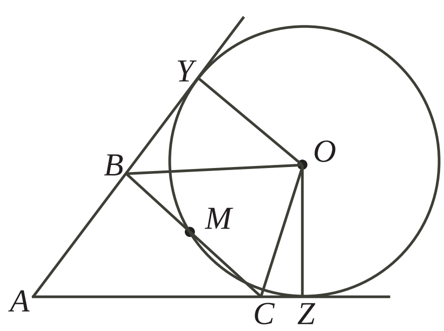
Suppose the centre of the circle is O and the circle has radius r.

Since the circle is tangent to AY and to AZ at Y and Z, respectively, then OY and OZ are perpendicular to AY and AZ.

Also, joining O to A bisects $\angle YAZ$ (since the circle is tangent to AY and AZ), so $\angle YAO=30^\circ$.

Thus, $AY=\sqrt{3}YO=\sqrt{3}r$. Also, $AZ=AY=\sqrt{3}r$.

Next, join O to B and to C.



Since $AB=10$, then $BY=AY-AB=\sqrt{3}r-10$.

Since $AC=10$, then $CZ=AZ-AC=\sqrt{3}r-16$.

Since $\triangle OBY$ is right-angled at Y, then

$$OB^2 = BY^2 + OY^2 = (\sqrt{3}r - 10)^2 + r^2$$

Since $\triangle OCZ$ is right-angled at Z, then

$$OC^2 = CZ^2 + OZ^2 = (\sqrt{3}r - 16)^2 + r^2$$

In $\triangle OBC$, since $BM = MC$, then $OB^2 + OC^2 = 2BM^2 + 2OM^2$. (See the end for a proof of this.)
Therefore,

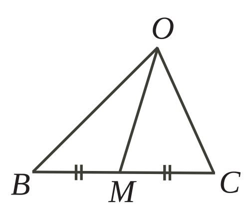
$$\begin{aligned}
(\sqrt{3}r - 10)^2 + r^2 + (\sqrt{3}r - 16)^2 + r^2 &= 2(7^2) + 2r^2 \\
3r^2 - 20\sqrt{3}r + 100 + r^2 + 3r^2 - 32\sqrt{3}r + 256 + r^2 &= 98 + 2r^2 \\
6r^2 - 52\sqrt{3}r + 258 &= 0 \\
3r^2 - 26\sqrt{3}r + 129 &= 0
\end{aligned}$$

As in Solution 1, $r = \dfrac{13\sqrt{3} + 2\sqrt{30}}{3}$, and so the minimum perimeter is

$$2\sqrt{3}r = \frac{26(3) + 4\sqrt{90}}{3} = 26 + 4\sqrt{10}$$

We could have noted, though, that since we want to find $2\sqrt{3}r$, then setting $z = \sqrt{3}r$, the equation $3r^2 - 26\sqrt{3}r + 129 = 0$ becomes $z^2 - 26z + 129 = 0$. Completing the square, we get $(z - 13)^2 = 40$, so $z = 13 \pm 2\sqrt{10}$, whence the perimeter is $26 + 4\sqrt{10}$ in similar way.

We must still justify that, in $\triangle OBC$, we have $OB^2 + OC^2 = 2BM^2 + 2OM^2$.



By the cosine law in $\triangle OBM$,

$$OB^2 = OM^2 + BM^2 - 2(OM)(BM)\cos(\angle OMB)$$

By the cosine law in $\triangle OCM$,

$$OC^2 = OM^2 + CM^2 - 2(OM)(CM)\cos(\angle OMC)$$

But $BM = CM$ and $\angle OMC = 180^\circ - \angle OMB$, so $\cos(\angle OMC) = -\cos(\angle OMB)$.
Therefore, our two equations become

$$\begin{aligned}
OB^2 &= OM^2 + BM^2 - 2(OM)(BM)\cos(\angle OMB) \\
OC^2 &= OM^2 + BM^2 + 2(OM)(BM)\cos(\angle OMB)
\end{aligned}$$

Adding, we obtain $OB^2 + OC^2 = 2OM^2 + 2BM^2$, as required.
(Notice that this result holds in any triangle with a median drawn in.)