



Canadian Mathematics Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

1998 Solutions

Euclid Contest

(Grade 12)

for the



NATIONAL BANK OF CANADA

Awards

1. (a) If one root of $x^2 + 2x - c = 0$ is $x = 1$, what is the value of c ?

Solution 1

If $x = 1$, by substituting, $c = 3$.

Solution 2

By division,

$$\begin{array}{r} x+3 \\ x-1 \overline{) x^2 + 2x - c} \\ \underline{x^2 - x} \\ 3x - c \\ \underline{3x - 3} \\ -c + 3 \end{array}$$

If the remainder is zero, $-c + 3 = 0$

$$c = 3.$$

- (b) If $2^{2x-4} = 8$, what is the value of x ?

Solution

$$2^{2x-4} = 2^3$$

$$\text{Therefore, } 2x - 4 = 3$$

$$x = \frac{7}{2}.$$

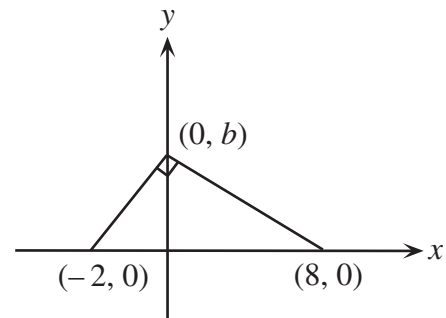
- (c) Two perpendicular lines with x -intercepts -2 and 8 intersect at $(0, b)$. Determine all values of b .

Solution 1

If the lines are perpendicular their slopes are negative reciprocals.

$$\text{Thus, } \frac{b}{-8} \times \frac{b}{2} = -1$$

$$b^2 = 16, b = \pm 4.$$



Solution 2

$$\text{Using Pythagoras, } [(b-0)^2 + (0-8)^2] + [(b-0)^2 + (0+2)^2] = 10^2$$

$$2b^2 = 32, b = \pm 4.$$

Solution 3

The vertices of the triangle represents three points on a circle with $(-2, 0)$ and $(8, 0)$ being the

coordinates of the end points of the diameter. This circle has centre $C(3, 0)$ and $r = 5$. The equation for this circle is $(x - 3)^2 + y^2 = 25$ and if we want to find the y -intercepts we let $x = 0$ which gives $b = \pm 4$.

2. (a) The vertex of $y = (x - 1)^2 + b$ has coordinates $(1, 3)$. What is the y -intercept of this parabola?

Solution

The vertex of parabola is $(1, b)$.

Therefore, $b = 3$.

The required equation is now $y = (x - 1)^2 + 3$.

For the y -intercept, let $x = 0$.

Thus, $y_{\text{int}} = (0 - 1)^2 + 3 = 4$.

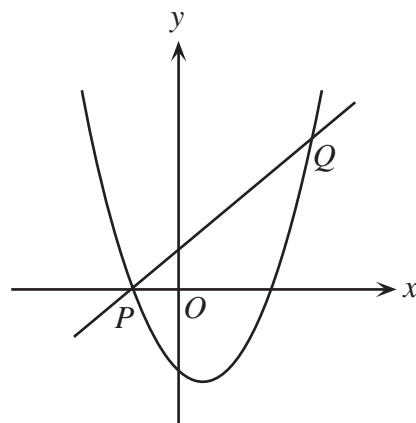
- (b) What is the area of $\triangle ABC$ with vertices $A(-3, 1)$, $B(5, 1)$ and $C(8, 7)$?

Solution

Drawing the diagram gives a triangle with a height of 6 and a base of 8 units.

The triangle has an area of 24 square units.

- (c) In the diagram, the line $y = x + 1$ intersects the parabola $y = x^2 - 3x - 4$ at the points P and Q . Determine the coordinates of P and Q .



Solution

Consider the system of equations $y = x + 1$, $y = x^2 - 3x - 4$.

Comparison gives $x + 1 = x^2 - 3x - 4$

$$x^2 - 4x - 5 = 0$$

$$(x - 5)(x + 1) = 0.$$

Therefore $x = 5$ or $x = -1$.

If $x = 5$, $y = 6$ and if $x = -1$, $y = 0$.

The required coordinates are $P(-1, 0)$ and $Q(5, 6)$.

3. (a) The graph of $y = m^x$ passes through the points $(2, 5)$ and $(5, n)$. What is the value of mn ?

Solution

Since $(2, 5)$ is on $y = m^x$, $5 = m^2$.

Since $(5, n)$ is on $y = m^x$, $n = m^5$.

So $mn = m(m^5) = m^6 = (m^2)^3 = 5^3 = 125$.

- (b) Jane bought 100 shares of stock at \$10.00 per share. When the shares increased to a value of $\$N$ each, she made a charitable donation of all the shares to the Euclid Foundation. She received a tax refund of 60% on the total value of her donation. However, she had to pay a tax of 20% on the increase in the value of the stock. Determine the value of N if the difference between her tax refund and the tax paid was \$1000.

Solution

Jane's charitable donation to the Euclid Foundation was $100N$ dollars.

Her tax refund was 60% of $100N$ or $60N$ dollars.

The increase in the value of her stock was $100(N - 10)$ or $(100N - 1000)$ dollars.

Jane's tax payment was 20% of $100N - 1000$ or $20N - 200$.

From the given, $60N - (20N - 200) = 1000$

Upon simplification, $40N = 800$

$$N = 20.$$

Therefore the value of N was 20.

4. (a) Consider the sequence $t_1 = 1$, $t_2 = -1$ and $t_n = \left(\frac{n-3}{n-1}\right)t_{n-2}$ where $n \geq 3$. What is the value of t_{1998} ?

Solution 1

Calculating some terms, $t_1 = 1$, $t_2 = -1$, $t_3 = 0$, $t_4 = \frac{-1}{3}$, $t_5 = 0$, $t_6 = \frac{-1}{5}$ etc.

By pattern recognition, $t_{1998} = \frac{-1}{1997}$.

Solution 2

$$\begin{aligned} t_{1998} &= \frac{1995}{1997} t_{1996} = \frac{1995}{1997} \times \frac{1993}{1995} t_{1994} \\ &= \frac{1995}{1997} \cdot \frac{1993}{1995} \cdot \frac{1991}{1993} \cdots \frac{3}{5} \cdot \frac{1}{3} t_2 \\ &= \frac{-1}{1997} \end{aligned}$$

- (b) The n th term of an arithmetic sequence is given by $t_n = 555 - 7n$.
 If $S_n = t_1 + t_2 + \dots + t_n$, determine the smallest value of n for which $S_n < 0$.

Solution 1

This is an arithmetic sequence in which $a = 548$ and $d = -7$.

Therefore, $S_n = \frac{n}{2}[2(548) + (n-1)(-7)] = \frac{n}{2}[-7n + 1103]$.

We now want $\frac{n}{2}(-7n + 1103) < 0$.

Since $n > 0$, $-7n + 1103 < 0$

$$n > 157\frac{4}{7}.$$

Therefore the smallest value of n is 158.

Solution 2

For this series we want, $\sum_{k=1}^n t_k < 0$, or $\sum_{k=1}^n (555 - 7k) < 0$.

Rewriting, $555n - 7\frac{(n)(n+1)}{2} < 0$

$$1110n - 7n^2 - 7n < 0$$

$$7n^2 - 1103n > 0$$

$$\text{or, } n > \frac{1103}{7}.$$

The smallest value of n is 158.

Solution 3

We generate the series as 548, 541, 534, ..., 2, -5, ..., -544, -551.

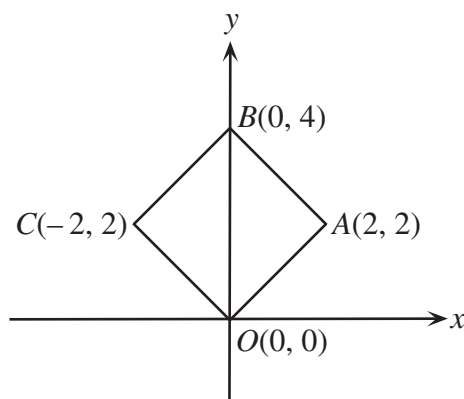
If we pair the series from front to back the sum of each pair is -3.

Including all the pairs 548 - 551, 541 - 544 and so on there would be 79 pairs which give a sum of -237.

If the last term, -551, were omitted we would have a positive sum.

Therefore we need all 79 pairs or 158 terms.

5. (a) A square $OABC$ is drawn with vertices as shown. Find the equation of the circle with largest area that can be drawn inside the square.



Solution

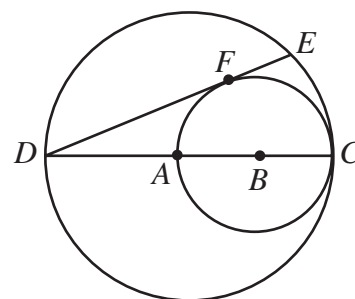
The square has a side length of $2\sqrt{2}$.

The diameter of the inscribed circle is $2\sqrt{2}$, so its radius is $\sqrt{2}$.

The centre of the circle is $(0, 2)$.

The required equation is $x^2 + (y - 2)^2 = 2$ or $x^2 + y^2 - 4y + 2 = 0$.

- (b) In the diagram, DC is a diameter of the larger circle centred at A , and AC is a diameter of the smaller circle centred at B . If DE is tangent to the smaller circle at F , and $DC = 12$, determine the length of DE .

**Solution**

Join B to F and C to E .

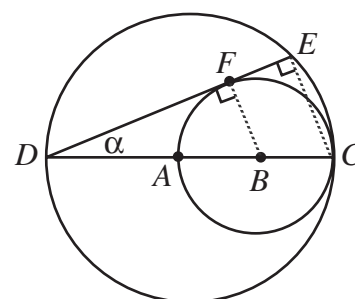
$FB \perp DE$ and DFE is a tangent.

Since DC is a diameter, $\angle DEC = 90^\circ$.

Thus $FB \parallel EC$.

By Pythagoras, $DF = \sqrt{9^2 - 3^2} = \sqrt{72}$.

Using similar triangles (or the side splitting theorem) we have,



OR

$$\frac{DE}{DF} = \frac{DC}{DB}$$

$$\frac{EC}{FB} = \frac{12}{9}$$

$$\frac{DE}{6\sqrt{2}} = \frac{4}{3}$$

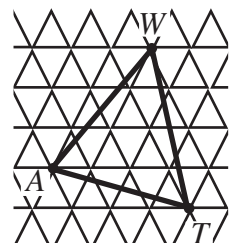
$$EC = \frac{4}{3}FB$$

$$DE = 8\sqrt{2} \text{ or } \sqrt{128}$$

$$EC = 4$$

By Pythagoras, $DE = 8\sqrt{2}$ or $\sqrt{128}$.

6. (a) In the grid, each small equilateral triangle has side length 1. If the vertices of $\triangle WAT$ are themselves vertices of small equilateral triangles, what is the area of $\triangle WAT$?



Solution 1

$$AT^2 = 1^2 + 4^2 - 2(1)(4)\cos 60^\circ = 13$$

Since $\triangle WAT$ is an equilateral triangle with a side of

length $\sqrt{13}$, its height will be $\frac{\sqrt{3}}{2}(\sqrt{13})$. The area of

$\triangle WAT$ is thus, $\frac{1}{2}\left[\left(\frac{\sqrt{3}}{2}\right)(\sqrt{13})\right]\sqrt{13} = \frac{13}{4}\sqrt{3}$. It is also possible to use the formula for the area of a triangle,

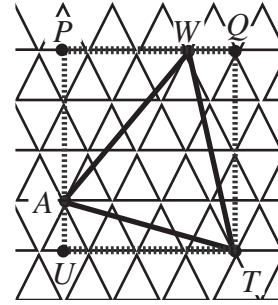
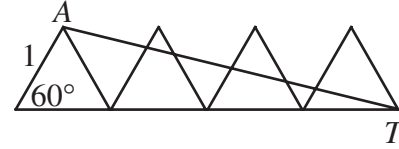
$$\text{Area} = \frac{1}{2}ab \sin c. \text{ Since the triangle is equilateral, area of } \triangle WAT = \frac{\sqrt{3}AT^2}{4} = \frac{13\sqrt{3}}{4}.$$

Solution 2

Since the small triangles have sides 1, they have a height of $\frac{\sqrt{3}}{2}$.

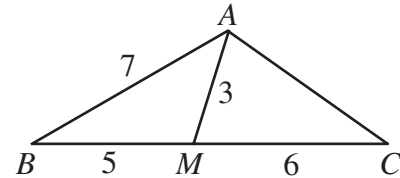
Consider rectangle $PQTU$.

Then



$$\begin{aligned} |\triangle WAT| &= |PQTU| - |\triangle APW| - |\triangle WQT| - |\triangle TUA| \\ &= (PQ)(QT) - \frac{1}{2}(AP)(PW) - \frac{1}{2}(WQ)(QT) - \frac{1}{2}(TU)(UA) \\ &= (3.5)(2\sqrt{3}) - \frac{1}{2}\left(\frac{3\sqrt{3}}{2}\right)(2.5) - \frac{1}{2}(1)(2\sqrt{3}) - \frac{1}{2}(3.5)\left(\frac{\sqrt{3}}{2}\right) \\ &= 7\sqrt{3} - \frac{15\sqrt{3}}{4} \\ &= \frac{13\sqrt{3}}{4} \end{aligned}$$

- (b) In $\triangle ABC$, M is a point on BC such that $BM = 5$ and $MC = 6$. If $AM = 3$ and $AB = 7$, determine the exact value of AC .



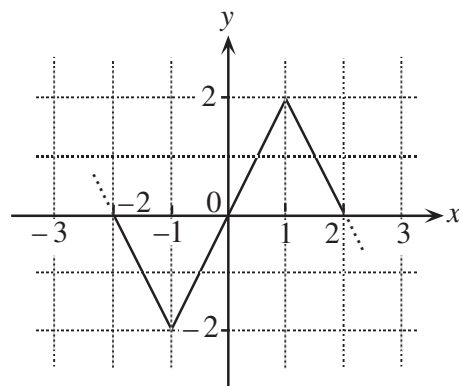
Solution

$$\text{From } \triangle ABM, \cos \angle B = \frac{3^2 - 7^2 - 5^2}{-2(7)(5)} = \frac{13}{14}.$$

$$\text{From } \triangle ABC, AC^2 = 7^2 + 11^2 - 2(7)(11)\left(\frac{13}{14}\right) = 27.$$

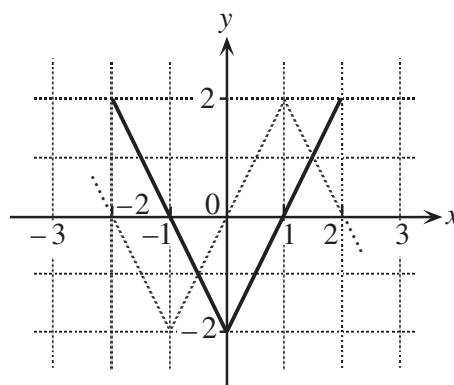
$$\text{Therefore, } AC = \sqrt{27}.$$

7. (a) The function $f(x)$ has period 4. The graph of one period of $y = f(x)$ is shown in the diagram. Sketch the graph of $y = \frac{1}{2}[f(x-1) + f(x+3)]$, for $-2 \leq x \leq 2$.



Solution 1

(a)	x	$f(x)$	$f(x-1)$	$f(x+3)$	$\frac{1}{2}[f(x-1) + f(x+3)]$
	-2	0	2	2	2
	-1	-2	0	0	0
	0	0	-2	-2	-2
	1	2	0	0	0
	2	0	2	2	2



Now plot the points and join them with straight line segments.

Solution 2

Since $f(x)$ has period 4, $f(x+3) = f(x-1)$.

Therefore, $y = \frac{1}{2}[f(x-1) + f(x+3)] = \frac{1}{2}[f(x-1) + f(x-1)] = f(x-1)$.

The required graph is that of $y = f(x-1)$ which is formed by shifting the given graph **1 unit** to the right.

- (b) If x and y are real numbers, determine all solutions (x, y) of the system of equations

$$x^2 - xy + 8 = 0$$

$$x^2 - 8x + y = 0.$$

Solution 1

Subtracting,

$$x^2 - xy + 8 = 0$$

$$x^2 - 8x + y = 0$$

$$\hline -xy + 8x + 8 - y = 0$$

$$8(1+x) - y(1+x) = 0$$

$$(8-y)(1+x) = 0$$

$$y = 8 \quad \text{or} \quad x = -1$$

If $y = 8$, both equations become $x^2 - 8x + 8 = 0$, $x = 4 \pm 2\sqrt{2}$.

If $x = -1$ both equations become $y + 9 = 0$, $y = -9$.

The solutions are $(-1, -9)$, $(4 + 2\sqrt{2}, 8)$ and $(4 - 2\sqrt{2}, 8)$.

Solution 2

If $x^2 - xy + 8 = 0$, $y = \frac{x^2 + 8}{x}$.

And $x^2 - 8x + y = 0$ implies $y = 8x - x^2$.

Equating, $\frac{x^2 + 8}{x} = 8x - x^2$

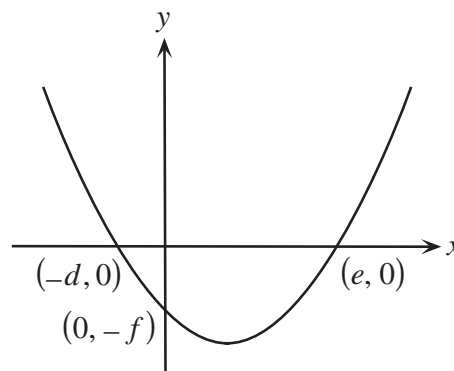
$$\text{or, } x^3 - 7x^2 + 8 = 0.$$

By inspection, $x = -1$ is a root.

By division, $x^3 - 7x^2 + 8 = (x + 1)(x^2 - 8x + 8)$.

As before, the solutions are $(-1, -9)$, $(4 \pm 2\sqrt{2}, 8)$.

8. (a) In the graph, the parabola $y = x^2$ has been translated to the position shown. Prove that $de = f$.



Solution

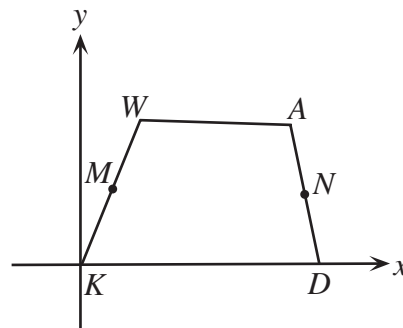
Since the given graph is congruent to $y = x^2$ and has x -intercepts $-d$ and e , its general form is $y = (x + d)(x - e)$.

To find the y -intercept, let $x = 0$. Therefore y -intercept $= -de$.

We are given that the y -intercept is $-f$.

Therefore $-f = -de$ or $f = de$.

- (b) In quadrilateral $KWAD$, the midpoints of KW and AD are M and N respectively. If $MN = \frac{1}{2}(AW + DK)$, prove that WA is parallel to KD .



Solution 1

Establish a coordinate system with $K(0, 0)$, $D(2a, 0)$ on the x -axes. Let W be $(2b, 2c)$ and A be $(2d, 2e)$.

Thus M is (b, c) and N is $(a + d, e)$.

KD has slope 0 and slope $WA = \frac{e-c}{d-b}$.

Since $MN = \frac{1}{2}(AW + DK)$

$$\begin{aligned} & \sqrt{(a+d-b)^2 + (e-c)^2} \\ &= \frac{1}{2} \left(2a + \sqrt{(2d-2b)^2 + (2e-2c)^2} \right) \\ &= \frac{1}{2} \left(2a + 2\sqrt{(d-b)^2 + (e-c)^2} \right) \end{aligned}$$

Squaring both sides gives,

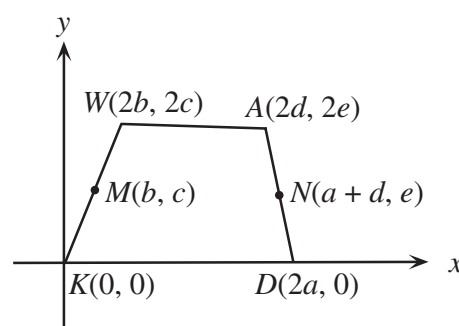
$$\begin{aligned} (a+d-b)^2 + (e-c)^2 &= a^2 + 2a\sqrt{(d-b)^2 + (e-c)^2} + (d-b)^2 + (e-c)^2 \\ a^2 + 2a(d-b) + (d-b)^2 &= a^2 + 2a\sqrt{(d-b)^2 + (e-c)^2} + (d-b)^2 \end{aligned}$$

Simplifying and dividing by $2a$ we have, $d-b = \sqrt{(d-b)^2 + (e-c)^2}$.

Squaring, $(d-b)^2 = (d-b)^2 + (e-c)^2$.

Therefore $(e-c)^2 = 0$ or $e = c$.

Since $e = c$ then slope of WA is 0 and $KD \parallel AW$.

**Solution 2**

Join A to K and call P the mid-point of AK .

Join M to P , N to P and M to N .

In $\triangle KAW$, P and M are the mid-points of KA and KW .

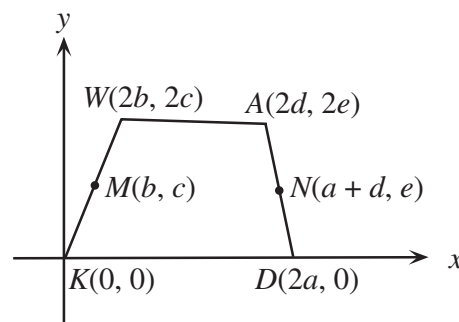
Therefore, $MP = \frac{1}{2}WA$.

Similarly in $\triangle KAD$, $PN = \frac{1}{2}KD$.

Therefore $MP + PN = MN$.

As a result M , P and N cannot form the vertices of a triangle but must form a straight line.

So if MPN is a straight line with $MP \parallel WA$ and $PN \parallel KD$ then $WA \parallel KD$ as required.

**Solution 3**

We are given that $\overrightarrow{AN} = \overrightarrow{ND}$ and $\overrightarrow{WM} = \overrightarrow{MK}$.

Using vectors,

$$(1) \quad \overrightarrow{MN} = \overrightarrow{MW} + \overrightarrow{WA} + \overrightarrow{AN} \quad (\text{from quad. } MWAN)$$

$$(2) \quad \overrightarrow{MN} = \overrightarrow{MK} + \overrightarrow{KD} + \overrightarrow{DN} \quad (\text{from quad. } KMND)$$

It is also possible to write, $\overrightarrow{MN} = -\overrightarrow{MW} + \overrightarrow{KD} - \overrightarrow{AN}$,

(3) (This comes from taking statement (2) and making appropriate substitutions.)

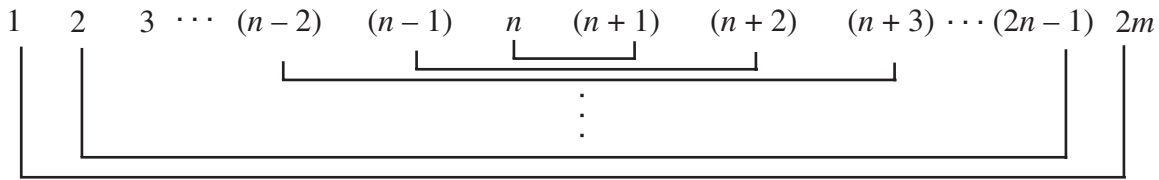
If we add (1) and (3) we find, $2\overrightarrow{MN} = \overrightarrow{WA} + \overrightarrow{KD}$.

But it is given that $2|\overrightarrow{MN}| = |\overrightarrow{AW}| + |\overrightarrow{DK}|$.

From these two previous statements, \overrightarrow{MN} must be parallel to \overrightarrow{WA} and \overrightarrow{KD} otherwise $2|\overrightarrow{MN}| < |\overrightarrow{AW}| + |\overrightarrow{DK}|$.

Therefore, $WA \parallel KD$.

9. Consider the first $2n$ natural numbers. Pair off the numbers, as shown, and multiply the two members of each pair. Prove that there is no value of n for which two of the n products are equal.



Solution 1

The sequence is $1(2n), 2(2n-1), 3(2n-2), \dots, k(2n-k+1), \dots, p(2n-p+1), \dots, n(n+1)$.

In essence we are asking the question, 'is it possible that $k(2n-k+1) = p(2n-p+1)$ where p and k are both less than or equal to n '?

$$k(2n-k+1) = p(2n-p+1) \quad (\text{supposing them to be equal})$$

$$2nk - k^2 + k = 2np - p^2 + p$$

$$p^2 - k^2 + 2nk - 2np + k - p = 0$$

$$(p-k)(p+k) + 2n(k-p) + (k-p) = 0$$

$$(p-k)[(p+k) - 2n - 1] = 0$$

$$(p-k)(p+k-2n-1) = 0$$

Since p and k are both less than or equal to n , it follows $p+k-2n-1 \neq 0$. Therefore $p=k$ and they represent the same pair. Thus the required is proven.

Solution 2

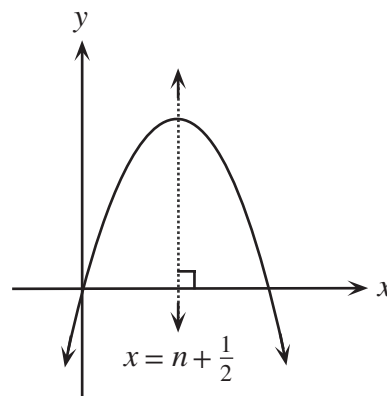
The products are $1(2n+1-1), 2(2n+1-2), 3(2n+1-3), \dots, n(2n+1-n)$.

Consider the function, $y = x(2n+1-x) = -x^2 + (2n+1)x = f(x)$.

The graph of this function is a parabola, opening down, with its vertex at $x = n + \frac{1}{2}$.

The products are the y-coordinates of the points on the parabola corresponding to $x = 1, 2, 3, \dots, n$. Since all the points are to the left of the vertex, no two have the same y-coordinate.

Thus the products are distinct.



Solution 3

The sum of these numbers is $\frac{2n(2n+1)}{2}$ or $n(2n+1)$.

Their average is $\frac{n(2n+1)}{2n} = n + \frac{1}{2}$.

The $2n$ numbers can be rewritten as,

$$n + \frac{1}{2} - \left(\frac{2n-1}{2}\right), \dots, n + \frac{1}{2} - \frac{3}{2}, n + \frac{1}{2} - \frac{1}{2}, n + \frac{1}{2} + \frac{1}{2}, n + \frac{1}{2} + \frac{3}{2}, \dots, n + \frac{1}{2} + \left(\frac{2n-1}{2}\right).$$

The product pairs, starting from the middle and working outward are

$$P_1 = \left(n + \frac{1}{2}\right)^2 - \frac{1}{4}$$

$$P_2 = \left(n + \frac{1}{2}\right)^2 - \frac{9}{4}$$

\vdots

$$P_n = \left(n + \frac{1}{2}\right)^2 - \left(\frac{2n-1}{2}\right)^2$$

Each of the numbers $\left(\frac{2k-1}{2}\right)^2$ is distinct for $k = 1, 2, 3, \dots, n$ and hence no terms of P_k are equal.

Solution 4

The sequence is $1(2n), 2(2n-1), 3(2n-2), \dots, n[2n-(n-1)]$.

This sequence has exactly n terms.

When the k th term is subtracted from the $(k+1)$ th term the difference is

$$(k+1)[2n-k] - k[2n-(k-1)] = 2(n-k). \text{ Since } n > k, \text{ this is a positive difference.}$$

Therefore each term is greater than the term before, so no two terms are equal.

10. The equations $x^2 + 5x + 6 = 0$ and $x^2 + 5x - 6 = 0$ **each** have integer solutions whereas only one of the equations in the pair $x^2 + 4x + 5 = 0$ and $x^2 + 4x - 5 = 0$ has integer solutions.
- (a) Show that if $x^2 + px + q = 0$ and $x^2 + px - q = 0$ **both** have integer solutions, then it is possible to find integers a and b such that $p^2 = a^2 + b^2$. (i.e. (a, b, p) is a Pythagorean triple).

- (b) Determine q in terms of a and b .

Solution

- (a) We have that $x^2 + px + q = 0$ and $x^2 + px - q = 0$ both have integer solutions.

For $x^2 + px + q = 0$, its roots are $\frac{-p \pm \sqrt{p^2 - 4q}}{2}$.

In order that these roots be integers, $p^2 - 4q$ must be a perfect square.

Therefore, $p^2 - 4q = m^2$ for some positive integer m .

Similarly for $x^2 + px - q = 0$, it has roots $\frac{-p \pm \sqrt{p^2 + 4q}}{2}$ and in order that these roots be integers $p^2 + 4q$ must be a perfect square.

Thus $p^2 + 4q = n^2$ for some positive integer n .

Adding gives $2p^2 = m^2 + n^2$ (with $n \geq m$ since $n^2 = p^2 + 4q \geq p^2 - 4q = m^2$)

And so $p^2 = \frac{1}{2}m^2 + \frac{1}{2}n^2 = \left(\frac{n+m}{2}\right)^2 + \left(\frac{n-m}{2}\right)^2$.

We note that m and n have the same parity since $m^2 = p^2 - 4q \equiv p^2 \pmod{2}$ and $n^2 \equiv p^2 + 4q \equiv p^2 \pmod{2}$.

Since $\frac{n+m}{2}$ and $\frac{n-m}{2}$ are positive integers then $p^2 = a^2 + b^2$ where $a = \frac{n+m}{2}$ and $b = \frac{n-m}{2}$.

- (b) From (a), $a = \frac{n+m}{2}$ and $b = \frac{n-m}{2}$ or $n = a+b$ and $m = a-b$.

$$\begin{aligned} \text{From before, } p^2 + 4q &= n^2 \\ 4q^2 &= n^2 - p^2 \\ &= (a+b)^2 - (a^2 + b^2) \end{aligned}$$

$$4q = 2ab.$$

Therefore, $q = \frac{ab}{2}$.