



The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2015 Euclid Contest

Wednesday, April 15, 2015
(in North America and South America)

Thursday, April 16, 2015
(outside of North America and South America)

Solutions

1. (a) Evaluating, $\frac{10^2 - 9^2}{10 + 9} = \frac{100 - 81}{19} = \frac{19}{19} = 1$.
 Alternatively, we could factor $10^2 - 9^2$ as a difference of squares to obtain

$$\frac{10^2 - 9^2}{10 + 9} = \frac{(10 + 9)(10 - 9)}{10 + 9} = 10 - 9 = 1$$

noting that $10 + 9$, which we divided from the numerator and denominator, is not equal to 0.

- (b) Since $\frac{x + 1}{x + 4} = 4$, then $x + 1 = 4(x + 4)$ and so $x + 1 = 4x + 16$ or $3x = -15$.

Therefore, $3x + 8 = -15 + 8 = -7$.

Alternatively, we could note that since $3x = -15$, then $x = -5$.

Thus, $3x + 8 = 3(-5) + 8 = -15 + 8 = -7$.

- (c) Since $f(x) = 2x - 1$, then $f(3) = 2(3) - 1 = 5$.

Therefore, $(f(3))^2 + 2(f(3)) + 1 = 5^2 + 2(5) + 1 = 25 + 10 + 1 = 36$.

Alternatively, we could note that since $f(x) = 2x - 1$, then

$$(f(x))^2 + 2(f(x)) + 1 = (f(x) + 1)^2 = (2x - 1 + 1)^2 = 4x^2$$

and so $(f(3))^2 + 2(f(3)) + 1 = 4(3^2) = 36$.

2. (a) Since $\sqrt{a} + \sqrt{a} = 20$, then $2\sqrt{a} = 20$ or $\sqrt{a} = 10$, and so $a = 10^2 = 100$.

- (b) Let the radius of the larger circle be r .

Since the radius of the smaller circle is 1, then its area is $\pi \cdot 1^2 = \pi$.

Since the area between the circles is equal to the area of the smaller circle, then the area of the larger circle is $\pi + \pi = 2\pi$.

Thus, $\pi r^2 = 2\pi$ or $r^2 = 2$. Since $r > 0$, then $r = \sqrt{2}$.

- (c) Since 30 students had an average mark of 80, then the sum of the marks of these 30 students was $30 \cdot 80 = 2400$.

After 2 students dropped the class, there were 28 students left. Their average mark was 82.

Thus, the sum of the marks of the remaining 28 students was $28 \cdot 82 = 2296$.

Therefore, the sum of the marks of the 2 students who dropped the class was $2400 - 2296$ or 104, and so their average mark was $\frac{104}{2} = 52$.

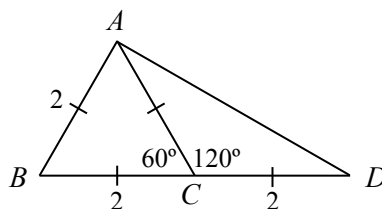
3. (a) *Solution 1*

Join AD .

Since $BC = CD$ and $BD = 4$, then $BC = CD = 2$. Also, $AB = BC = 2$.

Since $\triangle ABC$ is equilateral, then $\angle ABC = \angle ACB = 60^\circ$.

Since $\angle ACB = 60^\circ$, then $\angle ACD = 180^\circ - \angle ACB = 180^\circ - 60^\circ = 120^\circ$.



Since $AC = CD$, then $\triangle ACD$ is isosceles with $\angle CDA = \angle CAD$.

Each of these angles equals $\frac{1}{2}(180^\circ - \angle ACD) = \frac{1}{2}(180^\circ - 120^\circ) = 30^\circ$.

Since $\angle ABD = 60^\circ$ and $\angle ADB = 30^\circ$, then $\angle BAD = 90^\circ$ and $\triangle DBA$ is a 30° - 60° - 90° triangle.

Therefore, $AD = \sqrt{3}AB = 2\sqrt{3}$.

Solution 2

Join AD .

Since $BC = CD$ and $BD = 4$, then $BC = CD = 2$. Also, $AC = CD = 2$.

Since $\angle ACB = 60^\circ$, then $\angle ACD = 180^\circ - \angle ACB = 180^\circ - 60^\circ = 120^\circ$.

By the cosine law in $\triangle ACD$,

$$\begin{aligned} AD^2 &= AC^2 + CD^2 - 2(AC)(CD)\cos(\angle ACD) \\ &= 2^2 + 2^2 - 2(2)(2)\cos 120^\circ \\ &= 4 + 4 - 8\left(-\frac{1}{2}\right) \\ &= 12 \end{aligned}$$

Since $AD^2 = 12$ and $AD > 0$, then $AD = \sqrt{12} = 2\sqrt{3}$.

Solution 3

Join AD and drop a perpendicular from A to E on BC .

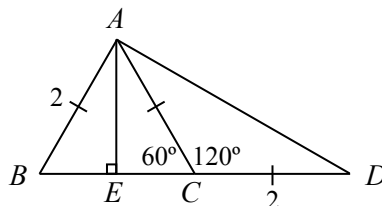
Since $BC = CD$ and $BD = 4$, then $BC = CD = 2$. Also, $AB = BC = 2$.

Since $\triangle ABC$ is equilateral, then $\angle ABC = \angle ACB = 60^\circ$.

Since $\angle ABC = 60^\circ$ and $\angle AEB = 90^\circ$, then $\triangle ABE$ is a 30° - 60° - 90° triangle.

Thus, $AE = \frac{\sqrt{3}}{2}AB = \sqrt{3}$.

Since $\angle ACB = 60^\circ$, then $\angle ACD = 180^\circ - \angle ACB = 180^\circ - 60^\circ = 120^\circ$.

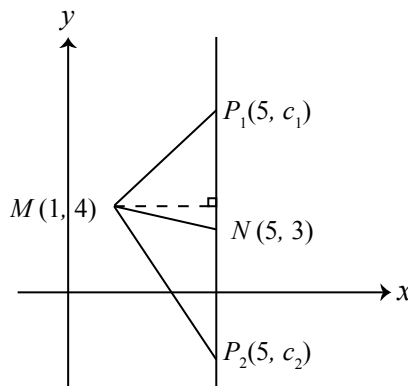


Since $AC = CD$, then $\triangle ACD$ is isosceles with $\angle CDA = \angle CAD$.

Each of these angles equals $\frac{1}{2}(180^\circ - \angle ACD) = \frac{1}{2}(180^\circ - 120^\circ) = 30^\circ$.

But $\triangle DAE$ is then a 30° - 60° - 90° triangle, so $AD = 2AE = 2\sqrt{3}$.

- (b) Points $N(5, 3)$ and $P(5, c)$ lie on the same vertical line. We can consider NP as the base of $\triangle MNP$. Suppose that the length of this base is b .
 The corresponding height of $\triangle MNP$ is the distance from $M(1, 4)$ to the line through N and P . Since M lies on the vertical line $x = 1$ and N and P lie on the vertical line $x = 5$, then the height is $h = 4$.



Since the area of $\triangle MNP$ is 14, then $\frac{1}{2}bh = 14$.

Since $h = 4$, then $\frac{1}{2}b(4) = 14$ or $2b = 14$ and so $b = 7$.

Therefore, $P(5, c)$ is a distance of 7 units away from $N(5, 3)$.

Since NP is a vertical line segment, then $c = 3 + 7$ or $c = 3 - 7$, and so $c = 10$ or $c = -4$.

The sum of these two values is $10 + (-4) = 6$.

(We could also have noted that, since the two values of c will be symmetric about $y = 3$, then the average of their values is 3 and so the sum of their values is $2 \cdot 3 = 6$.)

4. (a) To find the y -intercept, we set $x = 0$ and obtain

$$y = (-1)(-2)(-3) - (-2)(-3)(-4) = (-6) - (-24) = 18.$$

To find the x -intercepts, we first simplify using common factors:

$$y = (x-1)(x-2)(x-3) - (x-2)(x-3)(x-4) = (x-2)(x-3)((x-1) - (x-4)) = 3(x-2)(x-3)$$

To find the x -intercepts, we set $y = 0$ and obtain $3(x-2)(x-3) = 0$ which gives $x = 2$ or $x = 3$.

Therefore, the y -intercept is 18 and the x -intercepts are 2 and 3.

- (b) To find the points of intersection of the graphs with equations $y = x^3 - x^2 + 3x - 4$ and $y = ax^2 - x - 4$, we equate values of y and solve for x .

We want to find all values of a for which there are exactly two values of x which are solutions to $x^3 - x^2 + 3x - 4 = ax^2 - x - 4$.

Solving, we obtain

$$\begin{aligned} x^3 - x^2 + 3x - 4 &= ax^2 - x - 4 \\ x^3 - x^2 - ax^2 + 4x &= 0 \\ x^3 - (a+1)x^2 + 4x &= 0 \\ x(x^2 - (a+1)x + 4) &= 0 \end{aligned}$$

Therefore $x = 0$ or $x^2 - (a+1)x + 4 = 0$.

Note that $x = 0$ is not a solution to $x^2 - (a+1)x + 4 = 0$, since when $x = 0$ is substituted

into the left side, we obtain 4 and not 0.

Therefore, for there to be exactly two points of intersection between the two graphs, the quadratic equation $x^2 - (a+1)x + 4 = 0$ must have exactly one solution.

Setting the discriminant equal to 0 (to obtain a single root), we obtain $(a+1)^2 - 4(1)(4) = 0$ or $(a+1)^2 = 16$, which gives $a+1 = \pm 4$.

If $a+1 = 4$, then $a = 3$; if $a+1 = -4$, then $a = -5$.

Therefore, the values of a for which the graphs with equations $y = x^3 - x^2 + 3x - 4$ and $y = ax^2 - x - 4$ intersect at exactly two points are $a = 3$ and $a = -5$.

(We can check that $y = x^3 - x^2 + 3x - 4$ and $y = 3x^2 - x - 4$ intersect exactly when $x = 0$ and $x = 2$, and that $y = x^3 - x^2 + 3x - 4$ and $y = -5x^2 - x - 4$ intersect exactly when $x = 0$ and $x = -2$.)

5. (a) Suppose that $AB = AC = DE = x$.

Since $DB = 9$, then $AD = x - 9$.

Since $EC = 8$, then $AE = x - 8$.

By the Pythagorean Theorem in $\triangle ADE$,

$$\begin{aligned} AD^2 + AE^2 &= DE^2 \\ (x-9)^2 + (x-8)^2 &= x^2 \\ x^2 - 18x + 81 + x^2 - 16x + 64 &= x^2 \\ x^2 - 34x + 145 &= 0 \\ (x-5)(x-29) &= 0 \end{aligned}$$

Therefore, $x = 5$ or $x = 29$.

Since $x \geq 9$ (because $AB \geq DB = 9$), then $DE = 29$.

- (b) Since each list contains 6 consecutive positive integers and the smallest integers in the lists are a and b , then the positive integers in the first list are $a, a+1, a+2, a+3, a+4, a+5$ and the positive integers in the second list are $b, b+1, b+2, b+3, b+4, b+5$.

Note that $1 \leq a < b$.

We first determine the pairs (a, b) for which 49 will appear in the third list, then determine which of these pairs give a third list that contains no multiple of 64, and then finally keep only those pairs for which there is a number in the third list larger than 75.

The first bullet tells us that 49 is the product of an integer in the first list and an integer in the second list.

Since $49 = 7^2$ and 7 is prime, then these integers are either 1 and 49 or 7 and 7.

If 1 is in one of the lists, then either $a = 1$ or $b = 1$. Since $1 \leq a < b$, then it must be that $a = 1$.

If 49 is in the second list, then one of $b, b+1, b+2, b+3, b+4, b+5$ equals 49, and so $44 \leq b \leq 49$.

Therefore, for 1 and 49 to appear in the two lists, then (a, b) must be one of

$$(1, 49), (1, 48), (1, 47), (1, 46), (1, 45), (1, 44) .$$

If 7 appears in the first list, then one of $a, a+1, a+2, a+3, a+4, a+5$ equals 7, so $2 \leq a \leq 7$. Similarly, if 7 appears in the second list, then $2 \leq b \leq 7$.

Therefore, for 7 to appear in both lists, then, knowing that $a < b$, then (a, b) must be one of

$$(2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (3, 4), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 7), (5, 6), (5, 7), (6, 7) .$$

The second bullet tells us that no pair of numbers in the first and second lists have a product that is a multiple of 64.

Given that the possible values of a and b are 1, 2, 3, 4, 5, 6, 7, 44, 45, 46, 47, 48, 49, then the possible integers in the two lists are those integers from 1 to 12, inclusive, and from 44 to 54, inclusive. (For example, if the first number in one list is 7, then the remaining numbers in this list are 8, 9, 10, 11, 12.)

There is no multiple of 32 or 64 in these lists.

Thus, for a pair of integers from these lists to have a product that is a multiple of 64, one is a multiple of 4 and the other is a multiple of 16, or both are multiples of 8.

If $(a, b) = (1, 48), (1, 47), (1, 46), (1, 45), (1, 44)$, then 4 appears in the first list and 48 appears in the second list; these have a product of 192, which is $3 \cdot 64$.

If $(a, b) = (1, 49)$, there is a multiple of 4 but not of 8 in the first list, and a multiple of 4 but not of 8 in the second list, so there is no multiple of 64 in the third list.

If $(a, b) = (3, 4), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 7), (5, 6), (5, 7), (6, 7)$, then 8 appears in both lists, so 64 appears in the third list.

If $(a, b) = (2, 3), (2, 4), (2, 5), (2, 6), (2, 7)$, then there is no multiple of 8 or 16 in the first list and no multiple of 16 in the second list, so there is no multiple of 64 in the third list. Therefore, after considering the first two bullets, the possible pairs (a, b) are $(1, 49), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7)$.

The third bullet tells us that there is at least one number in the third list that is larger than 75.

Given the possible pairs (a, b) are $(1, 49), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7)$, the corresponding pairs of largest integers in the lists are $(6, 54), (7, 8), (7, 9), (7, 10), (7, 11), (7, 12)$.

The corresponding largest integers in the third list are the products of the largest integers in the two lists; these products are 324, 56, 63, 70, 77, 84, respectively.

Therefore, the remaining pairs (a, b) are $(1, 49), (2, 6), (2, 7)$.

Having considered the three conditions, the possible pairs (a, b) are $(1, 49), (2, 6), (2, 7)$.

6. (a) We are told that when a , b and c are the numbers in consecutive sectors, then $b = ac$. This means that if a and b are the numbers in consecutive sectors, then the number in the next sector is $c = \frac{b}{a}$. (That is, each number is equal to the previous number divided by the one before that.)

Starting with the given 2 and 3 and proceeding clockwise, we obtain

$$2, \quad 3, \quad \frac{3}{2}, \quad \frac{3/2}{3} = \frac{1}{2}, \quad \frac{1/2}{3/2} = \frac{1}{3}, \quad \frac{1/3}{1/2} = \frac{2}{3}, \quad \frac{2/3}{1/3} = 2, \quad \frac{2}{2/3} = 3, \quad \frac{3}{2}, \quad \dots$$

After the first 6 terms, the first 2 terms (2 and 3) reappear, and so the first 6 terms will repeat again. (This is because each term comes from the previous two terms, so when two consecutive terms reappear, then the following terms are the same as when these two consecutive terms appeared earlier.)

Since there are 36 terms in total, then the 6 terms repeat exactly $\frac{36}{6} = 6$ times.

Therefore, the sum of the 36 numbers is $6 \left(2 + 3 + \frac{3}{2} + \frac{1}{2} + \frac{1}{3} + \frac{2}{3} \right) = 6(2+3+2+1) = 48$.

- (b) We consider two cases: $x > -1$ (that is, $x + 1 > 0$) and $x < -1$ (that is, $x + 1 < 0$). Note that $x \neq -1$.

Case 1: $x > -1$

We take the given inequality $0 < \frac{x^2 - 11}{x + 1} < 7$ and multiply through by $x + 1$, which is positive, to obtain $0 < x^2 - 11 < 7x + 7$.

Thus, $x^2 - 11 > 0$ and $x^2 - 11 < 7x + 7$.

From the first, we obtain $x^2 > 11$ and so $x > \sqrt{11}$ or $x < -\sqrt{11}$.

Since $x > -1$, then $x > \sqrt{11}$. (Note that $-\sqrt{11} < -1$.)

From the second, we obtain $x^2 - 7x - 18 < 0$ or $(x - 9)(x + 2) < 0$. Thus, $-2 < x < 9$. (Since $y = x^2 - 7x - 18$ represents a parabola opening upwards, its y -values are negative between its x -intercepts.)

Since $x > -1$ and $-2 < x < 9$, then $-1 < x < 9$.

Since $x > \sqrt{11}$ and $-1 < x < 9$, then the solution in this case is $\sqrt{11} < x < 9$.

Case 2: $x < -1$

We take the given inequality $0 < \frac{x^2 - 11}{x + 1} < 7$ and multiply through by $x + 1$, which is negative, to obtain $0 > x^2 - 11 > 7x + 7$.

Thus, $x^2 - 11 < 0$ and $x^2 - 11 > 7x + 7$.

From the first, we obtain $x^2 < 11$ and so $-\sqrt{11} < x < \sqrt{11}$.

Since $x < -1$ and $-\sqrt{11} < x < \sqrt{11}$, then $-\sqrt{11} < x < -1$.

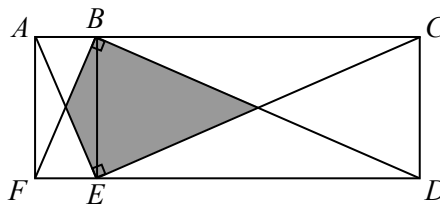
From the second, we obtain $x^2 - 7x - 18 > 0$ or $(x - 9)(x + 2) > 0$. Thus, $x < -2$ or $x > 9$. (Since $y = x^2 - 7x - 18$ represents a parabola opening upwards, its y -values are positive outside its x -intercepts.)

Since $x < -1$, we obtain $x < -2$.

Since $-\sqrt{11} < x < -1$ and $x < -2$, then the solution in this case is $-\sqrt{11} < x < -2$.

In summary, the values of x for which $0 < \frac{x^2 - 11}{x + 1} < 7$ those x with $-\sqrt{11} < x < -2$ and those x with $\sqrt{11} < x < 9$.

7. (a) Join BE .



Since $\triangle FBD$ is congruent to $\triangle AEC$, then $FB = AE$.

Since $\triangle FAB$ and $\triangle AFE$ are each right-angled, share a common side AF and have equal hypotenuses ($FB = AE$), then these triangles are congruent, and so $AB = FE$.

Now $BAFE$ has two right angles at A and F (so AB and FE are parallel) and has equal sides $AB = FE$ so must be a rectangle.

This means that $BCDE$ is also a rectangle.

Now the diagonals of a rectangle partition it into four triangles of equal area. (Diagonal AE of the rectangle splits the rectangle into two congruent triangles, which have equal area. The diagonals bisect each other, so the four smaller triangles all have equal area.)

Since $\frac{1}{4}$ of rectangle $ABEF$ is shaded and $\frac{1}{4}$ of rectangle $BCDE$ is shaded, then $\frac{1}{4}$ of the total area is shaded. (If the area of $ABEF$ is x and the area of $BCDE$ is y , then the total shaded area is $\frac{1}{4}x + \frac{1}{4}y$, which is $\frac{1}{4}$ of the total area $x + y$.)

Since $AC = 200$ and $CD = 50$, then the area of rectangle $ACDF$ is $200(50) = 10\,000$, so the total shaded area is $\frac{1}{4}(10\,000) = 2500$.

- (b) Suppose that the arithmetic sequence a_1, a_2, a_3, \dots has first term a and common difference d .

Then, for each positive integer n , $a_n = a + (n - 1)d$.

Since $a_1 = a$ and $a_2 = a + d$ and $a_1 \neq a_2$, then $d \neq 0$.

Since a_1, a_2, a_6 form a geometric sequence in that order, then $\frac{a_2}{a_1} = \frac{a_6}{a_2}$ or $(a_2)^2 = a_1 a_6$.

Substituting, we obtain

$$\begin{aligned} (a + d)^2 &= a(a + 5d) \\ a^2 + 2ad + d^2 &= a^2 + 5ad \\ d^2 &= 3ad \\ d &= 3a \quad (\text{since } d \neq 0) \end{aligned}$$

Therefore, $a_n = a + (n - 1)d = a + (n - 1)(3a) = (3n - 2)a$ for each $n \geq 1$.

Thus, $a_4 = (3(4) - 2)a = 10a$, and $a_k = (3k - 2)a$. (Note that $a_1 = (3(1) - 2)a = a$.)

For a_1, a_4, a_k to also form a geometric sequence then, as above, $(a_4)^2 = a_1 a_k$, and so

$$\begin{aligned} (10a)^2 &= (a)((3k - 2)a) \\ 100a^2 &= (3k - 2)a^2 \end{aligned}$$

Since $d \neq 0$ and $d = 3a$, then $a \neq 0$.

Since $100a^2 = (3k - 2)a^2$ and $a \neq 0$, then $100 = 3k - 2$ and so $3k = 102$ or $k = 34$.

Checking, we note that $a_1 = a$, $a_4 = 10a$ and $a_{34} = 100a$ which form a geometric sequence with common ratio 10.

Therefore, the only possible value of k is $k = 34$.

8. (a) First, we note that since k is a positive integer, then $k \geq 1$.

Next, we note that the given parabola passes through the point $(0, -5)$ as does the given circle. (This is because if $x = 0$, then $y = \frac{0^2}{k} - 5 = -5$ and if $(x, y) = (0, -5)$, then $x^2 + y^2 = 0^2 + (-5)^2 = 25$, so $(0, -5)$ satisfies each of the equations.)

Therefore, for every positive integer k , the two graphs intersect in at least one point.

If $y = -5$, then $x^2 + (-5)^2 = 25$ and so $x^2 = 0$ or $x = 0$. In other words, there is one point on both parabola and circle with $y = -5$, namely $(0, -5)$.

Now, the given circle with equation $x^2 + y^2 = 25 = 5^2$ has centre $(0, 0)$ and radius 5.

This means that the y -coordinates of points on this circle satisfy $-5 \leq y \leq 5$.

To find the other points of intersection, we re-write $y = \frac{x^2}{k} - 5$ as $ky = x^2 - 5k$ or $x^2 = ky + 5k$ and substitute into $x^2 + y^2 = 25$ to obtain

$$\begin{aligned}(ky + 5k) + y^2 &= 25 \\ y^2 + ky + (5k - 25) &= 0 \\ (y + 5)(y + (k - 5)) &= 0\end{aligned}$$

and so $y = -5$ or $y = 5 - k$.

(We note that since the two graphs intersect at $y = -5$, then $(y + 5)$ was going to be a factor of the quadratic equation $y^2 + ky + (5k - 25) = 0$. If we had not seen this, we could have used the quadratic formula.)

Therefore, for $y = 5 - k$ to give points on the circle, we need $-5 \leq 5 - k$ and $5 - k \leq 5$.

This gives $k \leq 10$ and $k \geq 0$.

Since k is a positive integer, the possible values of k to this point are $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

If $k = 1$, then $y = 5 - 1 = 4$. In this case, $x^2 + 4^2 = 25$ or $x^2 = 9$ and so $x = \pm 3$.

This gives the two points $(3, 4)$ and $(-3, 4)$ which lie on the parabola and circle.

Consider the three points $A(3, 4)$, $B(-3, 4)$ and $C(0, -5)$.

Now AB is horizontal with $AB = 3 - (-3) = 6$. (This is the difference in x -coordinates.)

The vertical distance from AB to C is $4 - (-5) = 9$. (This is the difference in y -coordinates.)

Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(6)(9) = 27$, which is a positive integer.

We now repeat these calculations for each of the other values of k by making a table:

k	y	$x = \pm\sqrt{25 - y^2}$	Base	Height	Area of triangle
1	4	± 3	$3 - (-3) = 6$	$4 - (-5) = 9$	27
2	3	± 4	$4 - (-4) = 8$	$3 - (-5) = 8$	32
3	2	$\pm\sqrt{21}$	$2\sqrt{21}$	7	$7\sqrt{21}$
4	1	$\pm\sqrt{24}$	$2\sqrt{24}$	6	$6\sqrt{24}$
5	0	± 5	10	5	25
6	-1	$\pm\sqrt{24}$	$2\sqrt{24}$	4	$4\sqrt{24}$
7	-2	$\pm\sqrt{21}$	$2\sqrt{21}$	3	$3\sqrt{21}$
8	-3	± 4	8	2	8
9	-4	± 3	6	1	3
10	-5	0			

When $k = 10$, we have $y = 5 - k = -5$ and $x = 0$ only, so there is only one point of intersection.

Finally, the values of k for which there are three points of intersection and for which the area of the resulting triangle is a positive integer are $k = 1, 2, 5, 8, 9$.

(b) Suppose that M is the midpoint of YZ .

Suppose that the centre of the smaller circle is O and the centre of the larger circle is P . Suppose that the smaller circle touches XY at C and XZ at D , and that the larger circle touches XY at E and XZ at F .

Join OC , OD and PE .

Since OC and PE are radii that join the centres of circles to points of tangency, then OC and PE are perpendicular to XY .

Join XM . Since $\triangle XYZ$ is isosceles, then XM (which is a median by construction) is an altitude (that is, XM is perpendicular to YZ) and an angle bisector (that is, $\angle MXY = \angle MXZ$).

Now XM passes through O and P . (Since XC and XD are tangents from X to the same circle, then $XC = XD$. This means that $\triangle XCO$ is congruent to $\triangle XDO$ by side-side-side. This means that $\angle OXC = \angle OXD$ and so O lies on the angle bisector of $\angle CXD$, and so O lies on XM . Using a similar argument, P lies on XM .)

Draw a perpendicular from O to T on PE . Note that OT is parallel to XY (since each is perpendicular to PE) and that $OCET$ is a rectangle (since it has three right angles).

Consider $\triangle XMY$ and $\triangle OTP$.

Each triangle is right-angled (at M and at T).

Also, $\angle YXM = \angle POT$. (This is because OT is parallel to XY , since both are perpendicular to PE .)

Therefore, $\triangle XMY$ is similar to $\triangle OTP$.

Thus, $\frac{XY}{YM} = \frac{OP}{PT}$.

Now $XY = a$ and $YM = \frac{1}{2}b$.

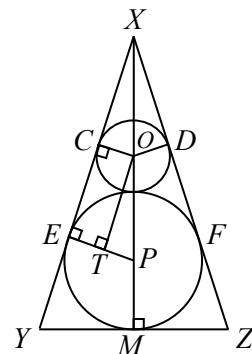
Also, OP is the line segment joining the centres of two tangent circles, so $OP = r + R$.

Lastly, $PT = PE - ET = R - r$, since $PE = R$, $ET = OC = r$, and $OCET$ is a rectangle.

Therefore,

$$\begin{aligned} \frac{a}{b/2} &= \frac{R+r}{R-r} \\ \frac{2a}{b} &= \frac{R+r}{R-r} \\ 2a(R-r) &= b(R+r) \\ 2aR - bR &= 2ar + br \\ R(2a-b) &= r(2a+b) \\ \frac{R}{r} &= \frac{2a+b}{2a-b} \quad (\text{since } 2a > b \text{ so } 2a-b \neq 0, \text{ and } r > 0) \end{aligned}$$

Therefore, $\frac{R}{r} = \frac{2a+b}{2a-b}$.



9. Using logarithm rules $\log(uv) = \log u + \log v$ and $\log(s^t) = t \log s$ for all $u, v, s > 0$, the first equation becomes

$$\begin{aligned}(\log x)(\log y) - 3 \log 5 - 3 \log y - \log 8 - \log x &= a \\(\log x)(\log y) - \log x - 3 \log y - \log 8 - \log 5^3 &= a \\(\log x)(\log y) - \log x - 3 \log y - \log(8 \cdot 125) &= a \\(\log x)(\log y) - \log x - 3 \log y - \log(1000) &= a \\(\log x)(\log y) - \log x - 3 \log y - 3 &= a\end{aligned}$$

Similarly, the second equation becomes

$$\begin{aligned}(\log y)(\log z) - 4 \log 5 - 4 \log y - \log 16 - \log z &= b \\(\log y)(\log z) - 4 \log y - \log z - 4 \log 5 - \log 16 &= b \\(\log y)(\log z) - 4 \log y - \log z - \log(5^4 \cdot 16) &= b \\(\log y)(\log z) - 4 \log y - \log z - \log(10\,000) &= b \\(\log y)(\log z) - 4 \log y - \log z - 4 &= b\end{aligned}$$

And the third equation becomes

$$\begin{aligned}(\log z)(\log x) - 4 \log 8 - 4 \log x - 3 \log 625 - 3 \log z &= c \\(\log z)(\log x) - 4 \log x - 3 \log z - 4 \log 8 - 3 \log 625 &= c \\(\log z)(\log x) - 4 \log x - 3 \log z - \log(8^4 \cdot 625^3) &= c \\(\log z)(\log x) - 4 \log x - 3 \log z - \log(2^{12} \cdot 5^{12}) &= c \\(\log z)(\log x) - 4 \log x - 3 \log z - 12 &= c\end{aligned}$$

Since each of the steps that we have made are reversible, the original system of equations is equivalent to the new system of equations

$$\begin{aligned}(\log x)(\log y) - \log x - 3 \log y - 3 &= a \\(\log y)(\log z) - 4 \log y - \log z - 4 &= b \\(\log z)(\log x) - 4 \log x - 3 \log z - 12 &= c\end{aligned}$$

Next, we make the substitution $X = \log x$, $Y = \log y$ and $Z = \log z$. (This is equivalent to saying $x = 10^X$, $y = 10^Y$ and $z = 10^Z$.)

This transforms the system of equations to the equivalent system

$$\begin{aligned}XY - X - 3Y - 3 &= a \\YZ - 4Y - Z - 4 &= b \\XZ - 4X - 3Z - 12 &= c\end{aligned}$$

We re-write the first of these three equations as $X(Y - 1) - 3Y - 3 = a$ and then as $X(Y - 1) - 3(Y - 1) - 6 = a$ and then as $(X - 3)(Y - 1) = a + 6$.

In a similar way, we re-write the second and third of these equations to obtain the equivalent system

$$\begin{aligned}(X - 3)(Y - 1) &= a + 6 \\(Y - 1)(Z - 4) &= b + 8 \\(X - 3)(Z - 4) &= c + 24\end{aligned}$$

Next, we make the substitution $p = X - 3$, $q = Y - 1$ and $r = Z - 4$. (This is equivalent to saying $X = p + 3$, $Y = q + 1$ and $Z = r + 4$, or $x = 10^{p+3}$, $y = 10^{q+1}$ and $z = 10^{r+4}$.)

This transforms the original system of equations into the equivalent system

$$\begin{aligned} pq &= a + 6 \\ qr &= b + 8 \\ pr &= c + 24 \end{aligned}$$

We again note that this system of equations is equivalent to the initial system of equations, and each solution of this system corresponds with a solution of the initial system.

- (a) Suppose that $a = -4$, $b = 4$ and $c = -18$.

Then the last version of the system is

$$\begin{aligned} pq &= 2 \\ qr &= 12 \\ pr &= 6 \end{aligned}$$

Multiplying the three equations together gives $p^2q^2r^2 = 2 \cdot 12 \cdot 6 = 144$.

Since $(pqr)^2 = 144$, then $pqr = \pm 12$.

Therefore, $r = \frac{pqr}{pq} = \frac{\pm 12}{2} = \pm 6$ and $p = \frac{pqr}{qr} = \frac{\pm 12}{12} = \pm 1$ and $q = \frac{pqr}{pr} = \frac{\pm 12}{6} = \pm 2$.

Therefore, the solutions to the last version of the system are $(p, q, r) = (1, 2, 6)$ and $(p, q, r) = (-1, -2, -6)$.

Converting back to the original variables, we see that the solutions to the original system when $(a, b, c) = (-4, 4, -18)$ are $(x, y, z) = (10^4, 10^3, 10^{10})$ and $(x, y, z) = (10^2, 10^{-1}, 10^{-2})$.

- (b) We consider the various possibilities for the product, $(a + 6)(b + 8)(c + 24)$, of the right sides of the equations in the final form of the system above: whether it is positive, negative or equal to 0.

Case 1: $(a + 6)(b + 8)(c + 24) < 0$

As in (a), we multiply the three equations together to obtain $(pqr)^2 = (a + 6)(b + 8)(c + 24)$. Since the left side is at least 0 and the right side is negative, then there are no solutions to the system of equations in this case.

Case 2: $(a + 6)(b + 8)(c + 24) > 0$

As in (a), we multiply the three equations together to obtain $(pqr)^2 = (a + 6)(b + 8)(c + 24)$. Since $(pqr)^2 = (a + 6)(b + 8)(c + 24)$ and $(a + 6)(b + 8)(c + 24) > 0$, then $pqr = \pm \sqrt{(a + 6)(b + 8)(c + 24)}$.

Since $(a + 6)(b + 8)(c + 24) > 0$, then $\sqrt{(a + 6)(b + 8)(c + 24)}$ is well-defined.

Also, since $(a + 6)(b + 8)(c + 24) > 0$, then each of $a + 6$, $b + 8$, $c + 24$ is non-zero, so we can divide by each of these quantities.

As we did in (a), we can solve to obtain

$$\begin{aligned} p &= \frac{pqr}{qr} = \frac{\pm \sqrt{(a + 6)(b + 8)(c + 24)}}{b + 8} \\ q &= \frac{pqr}{pr} = \frac{\pm \sqrt{(a + 6)(b + 8)(c + 24)}}{c + 24} \\ r &= \frac{pqr}{pq} = \frac{\pm \sqrt{(a + 6)(b + 8)(c + 24)}}{a + 6} \end{aligned}$$

Since $(a+6)(b+8)(c+24) > 0$, these are all valid fractions and there are exactly two triples (p, q, r) that are solutions and so two triples (x, y, z) that are solutions to the original system.

Case 3: $(a+6)(b+8)(c+24) = 0$

Suppose that exactly one of $a+6$, $b+8$ and $c+24$ equals 0.

Without loss of generality, suppose that $a+6 = 0$, $b+8 \neq 0$ and $c+24 \neq 0$.

Since $pq = a+6 = 0$, then $p = 0$ or $q = 0$.

In this case, either $qr = b+8$ or $pr = c+24$ will equal 0, which contradicts our assumption that neither $b+8$ nor $c+24$ is 0.

Therefore, it cannot be the case that exactly one of $a+6$, $b+8$ and $c+24$ equals 0.

Suppose next that exactly two of $a+6$, $b+8$ and $c+24$ equal 0.

Without loss of generality, suppose that $a+6 = b+8 = 0$ and $c+24 \neq 0$.

Since $pr = c+24 \neq 0$, then $p \neq 0$ and $r \neq 0$.

Since $pq = a+6 = 0$ and $qr = b+8 = 0$ and $p \neq 0$ and $r \neq 0$, then $q = 0$.

In this case, any triple (p, q, r) with $q = 0$ and $pr = c+24 \neq 0$ is a solution to the system of equations.

Thus, when $a+6 = b+8 = 0$ and $c+24 \neq 0$ (that is, $(a, b, c) = (-6, -8, c)$ with $c \neq -24$), each triple $(p, q, r) = \left(p, 0, \frac{c+24}{p}\right)$ with $p \neq 0$ is a solution to the system of equations.

Each of these solutions corresponds to a solution to the original system of equations in (x, y, z) , so if $(a, b, c) = (-6, -8, c)$ with $c \neq -24$, then there are infinite number of solutions to the system of equations.

Similarly, if $(a, b, c) = (-6, b, -24)$ with $b \neq -8$ (that is, if $p = a+6 = 0$ and $r = c+24 = 0$ but $q = b+8 \neq 0$) or $(a, b, c) = (a, -8, -24)$ with $a \neq -6$, then there are infinitely many solutions (x, y, z) to the original system of equations.

Finally, we must consider the case of $a+6 = b+8 = c+24 = 0$.

Here, we must solve the system of equations

$$pq = 0$$

$$qr = 0$$

$$pr = 0$$

Each triple $(p, q, r) = (0, 0, r)$ is a solution of this system and there are infinitely many such solutions. (This is not all of the solutions, but represents infinitely many solutions.) Therefore, when $(a, b, c) = (-6, -8, -24)$, there are also infinitely many solutions to the original system of equations.

Therefore, the system of equations has an infinite number of solutions (x, y, z) precisely when $(a, b, c) = (-6, -8, c)$ for some real number c or $(a, b, c) = (-6, b, -24)$ for some real number b or $(a, b, c) = (a, -8, -24)$ for some real number a or $(a, b, c) = (-6, -8, -24)$. (This last triple is in fact included in each of the previous three families of triples.)

10. (a) The subsets of C_4 are:

$$\begin{array}{cccccc} \{\} & \{1\} & \{2\} & \{3\} & \{4\} & \\ \{1, 2\} & \{1, 3\} & \{1, 4\} & \{2, 3\} & \{2, 4\} & \{3, 4\} \\ \{1, 2, 3\} & \{1, 2, 4\} & \{1, 3, 4\} & \{2, 3, 4\} & \{1, 2, 3, 4\} & \end{array}$$

(There are 16 such subsets including the empty set $\{\}$ and the complete set $C_4 = \{1, 2, 3, 4\}$.)

Consider the Furoni family $A = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$.

Each of the following subsets of C_4 is already an element of A : $\{1, 2\}, \{1, 3\}, \{1, 4\}$.

Each of the following subsets of C_4 is a subset of one or more of the elements of A : $\{\}, \{1\}, \{2\}, \{3\}, \{4\}$.

Each of the following subsets of C_4 has the property that one or more of the elements of A is a subset of it: $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}$.

Since a Furoni family of C_4 cannot contain two subsets of C_4 one of which is a subset of the other, none of the subsets in either of these two lists can be added to A to form a larger Furoni family.

This leaves the following subsets of C_4 to consider as possible elements to add to A : $\{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}$.

If $\{2, 3, 4\}$ is added to A to form $A' = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\}$, then A' is still a Furoni family of C_4 and none of $\{2, 3\}, \{2, 4\}, \{3, 4\}$ can be added, since each is a subset of $\{2, 3, 4\}$. Therefore, A' is a Furoni family of C_4 to which no other subset can be added. If any of $\{2, 3\}, \{2, 4\}, \{3, 4\}$ is added to A , then $\{2, 3, 4\}$ cannot be added (since each of these three two element sets is a subset of $\{2, 3, 4\}$) but each of the remaining two element sets can be still added without violating the conditions for being a Furoni family.

Thus, $A'' = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$ is a Furoni family of C_4 to which no other subset can be added.

Therefore, the two Furoni families of C_4 that contain all of the elements of A and to which no further subsets of C_4 can be added are

$$A' = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\} \quad A'' = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$$

(b) *Solution 1*

Suppose that n is a positive integer and F is a Furoni family of C_n that contains a_k elements that contain exactly k integers each, for each integer k from 0 to n , inclusive.

Consider each element E of F .

Each E is a subset of C_n . Suppose that a particular choice for E contains exactly k elements.

We use E to generate $k!(n-k)!$ permutations σ of the integers in $C_n = \{1, 2, 3, \dots, n\}$ by starting with a permutation α of the elements of E and appending a permutation β of the elements in C_n not in E .

Since there are k elements in E , there are $k!$ possible permutations α .

Since there are $n-k$ elements in C_n that are not in E , there are $(n-k)!$ possible permutations β .

Each possible α can have each possible β appended to it, so there are $k!(n-k)!$ possible permutations $\sigma = \alpha|\beta$. (The notation " $\alpha|\beta$ " means the permutation of C_n formed by writing out the permutation α (of the elements of E) followed by writing out the permutation β (of the elements of C_n not in E).)

Each of these $k!(n-k)!$ permutations generated by E is indeed different, since if two permutations $\sigma = \alpha|\beta$ and $\sigma' = \alpha'|\beta'$ are equal, then since α and α' are both permutations of the elements of E , then they have the same length and so $\alpha|\beta = \alpha'|\beta'$ means $\alpha = \alpha'$.

This then means that $\beta = \beta'$ and so the permutations started out the same.

We repeat this process for each of the elements E of F .

Since, for each k , there are a_k subsets of size k in F , then the total number of permutations that this generates is

$$a_0 0!(n-0)! + a_1 1!(n-1)! + \cdots + a_{n-1} (n-1)!(n-(n-1))! + a_n n!(n-n)!$$

If each of these permutations is different, then this total is at most $n!$, since this is the total number of permutations of the elements of C_n .

Is it possible that two elements E and G of F generate identical permutations of the elements of C_n in this way?

Suppose that two permutations $\sigma = \alpha|\beta$ (generated by E) and $\sigma' = \alpha'|\beta'$ (generated by G) are identical.

Suppose that E contains k elements and G contains k' elements.

Either $k \leq k'$ or $k' \leq k$ (or both, if they are equal).

Without loss of generality, suppose that $k \leq k'$.

Then the length of α (which is k) is less than or equal to the length of α' (which is k').

But $\alpha|\beta = \alpha'|\beta'$, so this means that the first k entries in α' are equal to the first k entries in α .

But the entries in α are the elements of E and the entries of α' are the elements of G , so this means that E is a subset of G , which cannot be the case. This is a contradiction.

Therefore, each of the permutations generated by each of the subsets of C_n contained in F is unique.

Therefore,

$$a_0 0!(n-0)! + a_1 1!(n-1)! + \cdots + a_{n-1} (n-1)!(n-(n-1))! + a_n n!(n-n)! \leq n!$$

Dividing both sides by $n!$, we obtain successively

$$\begin{aligned} a_0 0!(n-0)! + a_1 1!(n-1)! + \cdots + a_{n-1} (n-1)!(n-(n-1))! + a_n n!(n-n)! &\leq n! \\ a_0 \frac{0!(n-0)!}{n!} + a_1 \frac{1!(n-1)!}{n!} + \cdots + a_{n-1} \frac{(n-1)!(n-(n-1))!}{n!} + a_n \frac{n!(n-n)!}{n!} &\leq 1 \\ a_0 \frac{1}{\binom{n}{0}} + a_1 \frac{1}{\binom{n}{1}} + \cdots + a_{n-1} \frac{1}{\binom{n}{n-1}} + a_n \frac{1}{\binom{n}{n}} &\leq 1 \\ \frac{a_0}{\binom{n}{0}} + \frac{a_1}{\binom{n}{1}} + \cdots + \frac{a_{n-1}}{\binom{n}{n-1}} + \frac{a_n}{\binom{n}{n}} &\leq 1 \end{aligned}$$

as required.

Solution 2

Suppose that n is a positive integer and that F is a randomly chosen Furoni family of C_n .

Consider $L = \{\{\}, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, \dots, n\}\}$.

The probability that the intersection of L and F is non-empty is at most 1.

Note that since each element of L is a subset of all of those to its right in the listing of L , then at most one of the elements of L can be in F .

If k is an integer with $k \geq 0$, the probability that $\{1, 2, 3, \dots, k\}$ is an element of F is $\frac{a_k}{\binom{n}{k}}$, where a_k is the number of elements in F that contain exactly k integers:

There are $\binom{n}{k}$ subsets of C_n that contain exactly k integer.

The probability that any particular one of these subsets is $\{1, 2, 3, \dots, k\}$ equals $\frac{1}{\binom{n}{k}}$.

Since a_k of these subsets are in F , then the probability that one of these a_k subsets is $\{1, 2, 3, \dots, k\}$ equals $\frac{a_k}{\binom{n}{k}}$.

(Note that we use the convention that if $k = 0$, then $\{1, 2, 3, \dots, k\} = \{\}$.)

The probability that any of the elements of L is in F is the sum of the probability of each element being in F , since at most one of the elements in L is in F .

Therefore,

$$\frac{a_0}{\binom{n}{0}} + \frac{a_1}{\binom{n}{1}} + \dots + \frac{a_{n-1}}{\binom{n}{n-1}} + \frac{a_n}{\binom{n}{n}} \leq 1$$

as required.

(c) Set $M = \binom{n}{k}$ where $k = \frac{1}{2}n$ if n is even and $k = \frac{1}{2}(n-1)$ if n is odd.

Then $\binom{n}{r} \leq M$ for every integer r with $0 \leq r \leq n$. (Recall that the largest entries in Pascal's Triangle are the one or two entries in the middle of each row. We prove this algebraically at the end.)

From (b),

$$\frac{a_0}{\binom{n}{0}} + \frac{a_1}{\binom{n}{1}} + \dots + \frac{a_{n-1}}{\binom{n}{n-1}} + \frac{a_n}{\binom{n}{n}} \leq 1$$

Multiplying through by M , we obtain

$$a_0 \frac{M}{\binom{n}{0}} + a_1 \frac{M}{\binom{n}{1}} + \dots + a_{n-1} \frac{M}{\binom{n}{n-1}} + a_n \frac{M}{\binom{n}{n}} \leq M$$

Since M is at least as large as each binomial coefficient, then each of the fractions on the left side is larger than 1 and so

$$a_0 + a_1 + \dots + a_{n-1} + a_n \leq a_0 \frac{M}{\binom{n}{0}} + a_1 \frac{M}{\binom{n}{1}} + \dots + a_{n-1} \frac{M}{\binom{n}{n-1}} + a_n \frac{M}{\binom{n}{n}} \leq M$$

Therefore, the total number of elements in the Furoni family F , which is $a_0 + a_1 + \cdots + a_n$, is at most M .

Is it possible to find a Furoni family of size M ?

Yes – the $M = \binom{n}{k}$ subsets of C_n of size k form a Furoni family, since no two sets of the same size can be subsets of each other without being equal. Therefore, the largest Furoni family of C_n has size $\binom{n}{k}$ when $n = 2k$ or $n = 2k + 1$ for some non-negative integer k .

We now prove the algebraic result above.

First, we note that $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = \binom{n}{n-r}$.

Therefore, if $\binom{n}{r} \leq \binom{n}{k}$ for all $r \leq k$, then $\binom{n}{r} \leq \binom{n}{k}$ for all k , since if $s > k$, then $s = n - r$ for some $r \leq k$ and so $\binom{n}{s} = \binom{n}{r} \leq \binom{n}{k}$.

Suppose first that $n = 2k$ for some positive integer k .

We prove that $\binom{n}{r} \leq \binom{n}{k}$ for each integer r with $0 \leq r \leq k$:

Since $n = 2k$, then

$$\frac{\binom{n}{r}}{\binom{n}{k}} = \frac{\frac{(2k)!}{r!(2k-r)!}}{\frac{(2k)!}{k!k!}} = \frac{k!}{r!} \frac{k!}{(2k-r)!}$$

If $r = k - d$ for some non-negative integer d , then

$$\frac{k!}{r!} \frac{k!}{(2k-r)!} = \frac{k!k!}{(k-d)!(k+d)!} = \frac{k(k-1) \cdots (k-d+1)}{(k+1)(k+2) \cdots (k+d)} = \frac{k}{k+1} \frac{k-1}{k+2} \cdots \frac{k-d+1}{k+d}$$

Since the right side is the product of d non-negative fractions, each of which is smaller than 1, then their product is smaller than 1.

Thus, $\binom{n}{r} \leq \binom{n}{k}$ if $0 \leq r \leq k$.

Suppose next that $n = 2k + 1$ for some non-negative integer k .

We prove that $\binom{n}{r} \leq \binom{n}{k}$ for each integer r with $0 \leq r \leq k$:

Since $n = 2k + 1$, then

$$\frac{\binom{n}{r}}{\binom{n}{k}} = \frac{\frac{(2k+1)!}{r!(2k+1-r)!}}{\frac{(2k+1)!}{k!(k+1)!}} = \frac{k!}{r!} \frac{(k+1)!}{(2k+1-r)!}$$

If $r = k - d$ for some non-negative integer d , then

$$\begin{aligned} \frac{k!}{r!} \frac{(k+1)!}{(2k+1-r)!} &= \frac{k!(k+1)!}{(k-d)!(k+1+d)!} \\ &= \frac{k(k-1) \cdots (k-d+1)}{(k+2)(k+3) \cdots (k+1+d)} \\ &= \frac{k}{k+2} \frac{k-1}{k+3} \cdots \frac{k-d+1}{k+1+d} \end{aligned}$$

Since the right side is the product of d non-negative fractions, each of which is smaller than 1, then their product is smaller than 1.

Thus, $\binom{n}{r} \leq \binom{n}{k}$ if $0 \leq r \leq k$.

This completes our proof.