# Duke Math Meet 2011: Team Round

In the Team Round, the entire team of six students will have 20 minutes to answer the 10 problems. The team members may collaborate freely, but as with all rounds in the Duke Math Meet, only pencil and paper may be used. After 20 minutes, the team will submit a single sheet with its answers to each of the 10 problems. The team will be given 10-minute, 1-minute, and 15-second warnings. Each correct answer will add 2 points to the team's score.

1. How many primes $p < 100$ satisfy $p = a^2 + b^2$ for some positive integers $a$ and $b$?

2. For $a < b < c$, there exists exactly one Pythagorean triple such that $a + b + c = 2000$. Find $a + c - b$.

3. Five points lie on the surface of a sphere of radius 1 such that the distance between any two points is at least $\sqrt{2}$. Find the maximum volume enclosed by these five points.

4. $ABCDEF$ is a convex hexagon with $AB = BC = CD = DE = EF = FA = 5$ and $AC = CE = EA = 6$. Find the area of $ABCDEF$.

5. Joe and Wanda are playing a game of chance. Each player rolls a fair 11-sided die, whose sides are labeled with numbers $1, 2, \cdots, 11$. Let the result of the Joe's roll be $X$, and the result of Wanda's roll be $Y$. Joe wins if $XY$ has remainder 1 when divided by 11, and Wanda wins otherwise. What is the probability that Joe wins?

6. Vivek picks a number and then plays a game. At each step of the game, he takes the current number and replaces it with a new number according to the following rule: if the current number $n$ is divisible by 3, he replaces $n$ with $\frac{n}{3} + 2$, and otherwise he replaces $n$ with $\lfloor 3 \log_3 n \rfloor$. If he starts with the number $3^{2011}$, what number will he have after 2011 steps? Note that $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$.

7. Define a sequence $a_n$ of positive real numbers with $a_1 = 1$, and
$$a_{n+1} = \cfrac{4a_n^2 - 1}{-2 + \cfrac{4a_n^2 - 1}{-2 + \cfrac{4a_n^2 - 1}{-2 + \dots}}}$$
What is $a_{2011}$?

8. A set $S$ of positive integers is called *good* if for any $x, y \in S$ either $x = y$ or $|x - y| \geq 3$. How many subsets of $\{1, 2, 3, \dots, 13\}$ are good? Include the empty set in your count.

9. Find all pairs of positive integers $(a, b)$ with $a \leq b$ such that $10 \cdot \text{lcm}(a, b) = a^2 + b^2$. Note that $\text{lcm}(m, n)$ denotes the least common multiple of $m$ and $n$.

10. For a natural number $n$, $g(n)$ denotes the largest odd divisor of $n$. Find
$$g(1) + g(2) + g(3) + \cdots + g\left(2^{2011}\right).$$