Duke Math Meet 2011: Power Round

In the Power Round the entire team of six students will have 60 minutes to answer this series of proof- based questions. The team members may collaborate freely, but as with all rounds in the Duke Math Meet, only pencil/pen and paper may be used (no calculators of any kind). After 60 minutes the team will submit all solutions. Solutions to different-numbered questions must be on different sheets of paper. Cross out anything you do not want graded. Teams will be given 30-minute, 5-minute, and 1-minute warnings. **Teams may use results of previous problems to solve later problems, even if the team has not submitted solutions to those previous problems.** The number of points earned for each problem varies, but the total is 25 points. To receive full points, solutions to problems must be clear mathematical proofs.

Increasing Multiplicative Functions

During this round, we will let the natural numbers be the set $\{1, 2, 3, \cdots\}$. Given two natural numbers m and n, the *greatest common divisor* of m and n, written $\gcd(m, n)$, is the largest natural number g such that g divides both m and n. For example, the greatest common divisor of 20 and 50 is 10. You should take careful note of the fact that $\gcd(m, n) = \gcd(m \pm n, n)$ for all integers m, n.

1. (0.5 points) If p and q are two distinct prime numbers, what is $\gcd(p, q)$? (Be sure to justify your answer.)

A pair of natural numbers m and n is called *relatively prime* if $\gcd(m, n) = 1$. For example, 15 and 28 are relatively prime, but 20 and 50 are *not* relatively prime.

Consider a function f that takes a natural number as input and returns a real number. If for any relatively prime natural numbers m and n, we have $f(mn) = f(m)f(n)$, then f is called a *multiplicative* function. If $f(mn) = f(m)f(n)$ for all natural numbers m and n, regardless of whether or not they are relatively prime, then f is called a *totally multiplicative* function. Note that a totally multiplicative function is, by definition, multiplicative.

You may assume that from this point forward, all functions discussed are not always zero.

2. (0.5 points) If f is multiplicative, what must $f(1)$ be equal to?

3. (1.5 points) Let $t(n)$ be the number of natural numbers that divide n, including 1 and n. Is $t(n)$ multiplicative? Is $t(n)$ totally multiplicative?

A function f is called an *increasing* function if for any natural numbers m and n with $m < n$, we have $f(m) \leq f(n)$. The function f is called *decreasing* if $f(m) \leq f(n)$ whenever $m > n$. As it turns out, increasing multiplicative functions (which we will henceforth abbreviate IMFs) have very special properties. In particular, while they must be increasing, they cannot grow very quickly. Let's try and make this statement more precise.

For a prime p and an IMF f, define the p-*growth factor of* f to be the greatest real number L such that $f(n + p) \geq Lf(n)$ for all natural numbers n that are relatively prime to p. We'll write this as $G(f, p)$. (Note that this is well-defined, as an IMF is never zero on the natural numbers.)

4. (0.5 points) Why must $G(f, p) \geq 1$ for all IMFs f and primes p?

5. (1 point) Let n be a natural number that is relatively prime to p. Let f be an IMF, and let k be a natural number. Show that

$$f(n + kp) \geq [G(f, p)]^k f(n).$$

6. a. (1.5 points) Using the same notation, show that $[G(f, p)]^k \leq f(2)$ for all natural numbers k. (Hint: choose n and k such that n is odd with $n > kp$.)

   b. (0.5 points) Explain why $G(f, p) = 1$ for all IMFs f and primes p.

Now that we know that all IMFs grow rather slowly, let's try and prove a stronger statement. Suppose that we are given an unknown multiplicative function f and told $f(p^k)$ for all primes p and natural numbers k. Then we can calculate $f(n)$ for any natural number n. However, if we are told that f is totally multiplicative, then we only need to be told $f(p)$ for all primes p. Let's see if we can prove a similar result about prime powers and IMFs.

7. a. (1.5 points) Let n be a natural number that is relatively prime to p, and let k be a natural number. Let f be an IMF. Show that

$$\frac{f(p^{k+1})}{f(p^k)} \leq \frac{f(np + 1)}{f(n)} \leq f(p)\frac{f(n + p)}{f(n)}.$$

   b. (0.5 points) Letting n, p, k, f be defined as above, and supposing that $n > p$, show that

$$f(p)\frac{f(n - p)}{f(n)} \leq \frac{f(np - 1)}{f(n)} \leq \frac{f(p^{k+1})}{f(p^k)}.$$

   c. (2.5 points) Show that

$$\frac{f(p)}{G(f, p)} \leq \frac{f(p^{k+1})}{f(p^k)} \leq f(p)G(f, p).$$

   d. (1 point) Explain why an IMF must be totally multiplicative.

8. (2.5 points) Show that if f is an IMF with $f(2) = 3$, then $f(n)$ is not always a natural number. (Hint: try calculating $f(n)$ for small values of n.)

9. a. (1 point) Show that if $0 < \alpha < 1$ is a real number, then there exist natural numbers A and B such that $\alpha B < A < B$.

   b. (5 points) Show that if f is an IMF, then there exists some fixed real number $\alpha \geq 0$ such that for all natural numbers n we have

$$f(n) = n^\alpha.$$

10. a. (1 point) Show that a decreasing multiplicative function must return only nonnegative values.

   b. (4 points) Find all of the decreasing multiplicative functions.