DUKE MATH MEET 2010 – POWER ROUND
CONDITIONAL PROBABILITY AND THE GAME OF CLUE

In the Power Round, each team of six students will have 60 minutes to answer this series of proof-based questions. Team members may collaborate freely in solving these questions. Only pencil and paper may be used during this round. After 60 minutes, each team will submit all solutions or solution attempts. Solutions to different questions must be written on different sheets of paper. Anything written on solution sheets that is not to be graded (scratch work, etc...) must be crossed or marked out. Teams will be given 30-minute, 5-minute, and 1-minute warnings. **Teams may use results of previous problems to solve later problems, even if they have not solved those previous problems.** Each problem is worth 3 points. If a question contains k parts, then each part is worth $\frac{3}{k}$ points. Partial credit may be awarded for partial solutions.

Some of the questions on this power round ask for a numerical answer (a probability) — this answer should be justified with a clear logical explanation; solutions with a correct answer but no proof will receive no credit.

Conditional Probability

We denote the probability of event A happening as $P(A)$. The *conditional probability of A given B*, or $P(A \mid B)$, is defined to be $P(A \,\text{and}\, B)/P(B)$ (assuming $P(B) > 0$); it can be interpreted as the probability of event A happening, given that we know the event B happens.

For example, suppose we're rolling a die. Let A be the event in which the die lands on a 4, 5 or 6; let B be the event in which the die lands on an even number. Then $P(A \,\text{and}\, B) = \frac{1}{3}$, $P(B) = \frac{1}{2}$, and therefore $P(A \mid B) = \frac{2}{3}$. This makes sense: suppose we know that event B has occurred — then the die must have landed on either 2, 4, or 6. Of these outcomes, only 4 and 6 coincide with event A; this represents 2 of the 3 possibilities, so the conditional probability is $\frac{2}{3}$.

1. Bayes's Law states that if $P(A)$ and $P(B)$ are both nonzero, then the following is true:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

   Prove Bayes's Law using the definition of conditional probability.

2. Show that $P(A \,\text{and}\, B \mid C) = P(B \mid C) \cdot P(A \mid B \,\text{and}\, C)$, assuming that each event or set of events occurs with nonzero probability.

The Game of Clue

We now consider a simplified version of the game of Clue. A murder was committed by Amy, Ben, or Chime (denoted A, B, and C, respectively) using either a xylophone, yo-yo, or zinc oxide (denoted x, y, and z, respectively). Two players compete to be the first to identify the murderer and the murder weapon. Three cards labeled A, B, and C are shuffled, then one is dealt to each player; the remaining card is put into a sealed envelope. The same is done with three more cards labeled x, y, and z. The cards in the sealed envelope identify the murderer and his weapon of choice. Each of the two players is holding a card identifying a person and a card identifying a murder weapon, which they can exclude from suspicion.

The players take turns, on which they may either make an accusation, or suggest a hypothesis. Both of these are given in the form "I think that *person* committed the murder using *weapon*." In the case of a hypothesis, the other player must, if possible, reveal a card to disprove the hypothesis. (For example, if Steve hypothesizes that A committed the murder with z and his opponent Thomas holds card A, then Thomas must reveal card A to disprove Steve's hypothesis. If Thomas holds A and z, then he may choose to reveal either card.) An accusation ends the game: the player who made it wins if it is correct, and loses otherwise.

You are playing this game against your friend Thomas. Assume that Thomas will always play optimally; i.e., with the information that he is given, he will try to maximize his probability of winning.

3. You are holding the cards A and x, and you make the hypothesis "Ben committed the murder with a yo-yo." To disprove this hypothesis, your friend reveals the card B. What is the probability that the murder was committed with a yo-yo?

4. Thomas knows nothing about your cards. He assumes that your hypothesis from the previous problem was chosen randomly from all nine possibilities (maybe he saw you rolling dice to pick it). He decides to assess the probability that you would be right if you made a best-guess accusation on your next turn, assuming you get no additional information.

   (a) What should he get, if he is holding the cards B and y?

   (b) What should he get, if he is holding the cards B and z?

   (c) In either case, should Thomas make an accusation on his turn? Why or why not?

5. Suppose you decide to choose a random hypothesis for your first move, as in the previous problem, and Thomas knows this. Describe your optimal strategy for all subsequent moves, and find the probability that you will win using this strategy.

6. Show that the following first-move strategies result in a lower probability of winning than the random-hypothesis strategy (assuming that Thomas knows the strategy you are using, but not the cards in your hand):

   (a) Choosing a random hypothesis that contains no cards in common with your hand — for instance, if your cards are A and x, the hypothesis would be randomly chosen from the set $\{(B, y), (B, z), (C, y), (C, z)\}$.

   (b) Choosing a random hypothesis that contains exactly one card in common with your hand — for instance, if your cards are A and x, the hypothesis would be randomly chosen from the set $\{(A, y), (A, z), (B, x), (C, x)\}$.

7. Give, with justification, a first-move strategy that yields a better probability of winning than choosing randomly from the set of all nine hypotheses. Calculate the probability that you will win using this strategy. As in the previous problem, we assume that Thomas knows the strategy you are using, but not the cards in your hand.