DUKE MATH MEET 2009: DEVIL ROUND

In the Devil Round, all teams will be broken apart and re-assembled randomly to form new teams of about 10 people each. There will be 4 sets of 3 problems each and the last set of one problem for a total of 13 problems. At the start of the Devil Round, one person from each team will run to the front of the room and grab the first set of problems and run back to solve them with their team. When the team wishes to submit the answers to a set, one person from the team will run to the front of the room to submit their answers and grab the next set of problems to take back to their team. There will be a combined total time of 15 minutes to solve all 13 problems; awards will be given to the fastest team that obtains the highest score.

Devil Round Problems 1, 2, and 3

1. Find all positive integers n such that $n^3 - 14n^2 + 64n - 93$ is prime.
2. Let a, b, c be real numbers such that $0 \leq a, b, c \leq 1$. Find the maximum value of
$$\frac{a}{1+bc} + \frac{b}{1+ac} + \frac{c}{1+ab}.$$
3. Find the maximum value of the function $f(x, y, z) = 4x + 3y + 2z$ on the ellipsoid
$$16x^2 + 9y^2 + 4z^2 = 1.$$

Devil Round Problems 4, 5, and 6

4. Let $x_1, \ldots, x_n$ be numbers such that $x_1 + \cdots + x_n = 2009$. Find the minimum value of $x_1^2 + \cdots + x_n^2$ (in term of n).
5. Find the number of odd integers between 1000 and 9999 that have at least 3 distinct digits.
6. Let $A_1, A_2, \ldots, A_{2^n-1}$ be all the possible nonempty subsets of $\{1, 2, 3, \ldots, n\}$. Find the maximum value of $a_1 + a_2 + \ldots + a_{2^n-1}$ where $a_i \in A_i$ for each $i = 1, 2, \ldots, 2^n - 1$.

Devil Round Problems 7, 8, and 9

7. Find the rightmost digit when 41^{2009} is written in base 7.
8. How many integral ordered triples (x, y, z) satisfy the equation $x+y+z = 2009$, where $10 \leq x < 31$, $100 < z < 310$ and $y \geq 0$.
9. Scooby has a fair six-sided die, labeled 1 to 6, and Shaggy has a fair twenty-sided die, labeled 1 to 20. During each turn, they both roll their own dice at the same time. They keep rolling the die until one of them rolls a 5. Find the probability that Scooby rolls a 5 before Shaggy does.

Devil Round Problems 10, 11, and 12

10. Let $N = 1A323492110877$ where A is a digit in the decimal expansion of N. Suppose N is divisible by 7. Find A.
11. Find all solutions (x, y) of the equation $\tan^4(x+y)+\cot^4(x+y) = 1-2x-x^2$, where $-\frac{\pi}{2} \leq x, y \leq \frac{\pi}{2}$.
12. Find the remainder when $\displaystyle\sum_{k=1}^{50} k!(k^2 + k - 1)$ is divided by 1008.

13. The *devil set* of a positive integer n, denoted $D(n)$, is defined as follows:

 (1) For every positive integer n, $n \in D(n)$.

 (2) If n is divisible by m and $m < n$, then for every element $a \in D(m)$, a^3 must be in $D(n)$.

 Furthermore, call a set S *scary* if for any $a, b \in S$, $a < b$ implies that b is divisible by a. What is the least positive integer n such that $D(n)$ is scary and has at least 2009 elements?