DUKE MATH MEET 2008: TIEBREAKER ROUND

These rules are not the same as the ones for 2007!

The Tiebreaker Round is given to students who give an outstanding performance on the individual round. The students with the highest score in the individual round are invited to the front of the auditorium, where they will be seated in desks facing the audience.

When time is called, each of the participants will be given the first question in the tiebreaker round. The participants will attempt to solve the tiebreaker round question individually. When the participant wishes to submit an answer, he or she will give it to any of the proctors. If the answer is correct, the time it took the student to solve the question will be recorded, otherwise the student may resume working on the problem. Students may submit as many answers as they wish.

If there are any students still working on the question after four minutes, they will be asked to stop. Rankings will be determined based first on the score on the individual round, then followed by the time required to submit a correct answer to the first question in the tiebreaker round. If the tie still has not been resolved, then the students will receive the second problem in the tiebreaker round, and the same rules will apply as before.

As all this is going on, the current problem will be projected at the front of the auditorium, so that members of the audience may attempt to solve the problem. The audience is forbidden to give hints or otherwise make any comments regarding the problems.

Tiebreaker Round

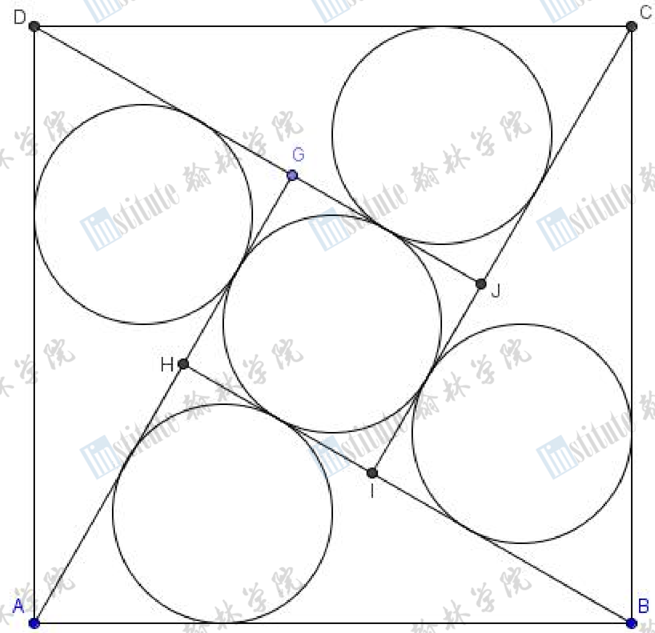
1. (See the diagram below.) $ABCD$ is a square. Points G, H, I, and J are chosen in the interior of $ABCD$ so that:

(i) H is on $\overline{AG}$, I is on $\overline{BH}$, J is on $\overline{CI}$, and G is on $\overline{DJ}$;

(ii) $\Delta ABH \cong \Delta BCI \cong \Delta CDJ \cong \Delta DAG$; and

(iii) the radii of the inscribed circles of ΔABH, ΔBCI, ΔCDJ, ΔDAK, and $GHIJ$ are all the same.

What is the ratio of $\overline{AB}$ to $\overline{GH}$?



2. The three solutions r_1, r_2, and r_3 of the equation

$$x^3 + x^2 - 2x - 1 = 0$$

can be written in the form $2\cos(k_1\pi)$, $2\cos(k_2\pi)$, and $2\cos(k_3\pi)$ where $0 \le k_1 < k_2 < k_3 \le 1$. What is the ordered triple (k_1, k_2, k_3)?

3. $\mathcal{P}$ is a convex polyhedron, all of whose faces are either triangles or decagons (10-sided polygon), though not necessarily regular. Furthermore, at each vertex of $\mathcal{P}$ exactly three faces meet. If $\mathcal{P}$ has 20 triangular faces, how many decagonal faces does $\mathcal{P}$ have?

4. P_1 is a parabola whose line of symmetry is parallel to the x-axis, has $(0, 1)$ as its vertex, and passes through $(2, 2)$.

P_2 is a parabola whose line of symmetry is parallel to the y-axis, has $(1, 0)$ as its vertex, and passes through $(2, 2)$.

Find all four points of intersection between P_1 and P_2.